22C:44 Homework 2 Solution

1. (a) Use the Master Theorem. So \( a = 2, b = 5, n^{\log_b a} = n^{\log_b 2} = n^{0.430677} \). Also \( f(n) = n^{0.5} \). Therefore, there exists \( \epsilon > 0 \) such that \( f(n) = \Omega(n^{\log_b a + \epsilon}) \). Also \( af(n/b) = 2f(n/5) = 2\sqrt{n/5} = 0.894427\sqrt{n} \). Hence the regularity condition is also satisfied. Applying Part (3) of the Master Theorem we get \( T(n) = \Theta(\sqrt{n}) \).

(b) Use the Iteration Method. Then we see that
\[
T(n) = 1/n + 2/n + 4/n + \cdots + 2^k/n + \Theta(1),
\]
where \( 2^k < n \) and \( 2^{k+1} \geq n \). Hence, \( T(n) = 1/n(2^{k+1} - 1) + \Theta(1) \). Since \( 2^{k+1} \leq 2n \), we get
\[
\frac{n - 1}{n} + \Theta(1) < T(n) \leq \frac{2n - 1}{n} + \Theta(1).
\]
Hence \( T(n) = \Theta(1) \).

Alternately, use the Master Theorem. So \( a = 1, b = 2, n^{\log_b a} = n^0 = 1, f(n) = 1/n = n^{-1} \). Therefore, there exists \( \epsilon > 0 \) such that \( f(n) = O(n^{\log_b a - \epsilon}) \) and by using Part (1) of the Master Theorem we get \( T(n) = \Theta(1) \).

(c) Guess that \( c3^n \leq T(n) \leq c'3^n \) for all \( n \geq 1 \). Choose \( c = \min\{T(1)/3, T(2)/9\} \) and \( c' = \max\{T(1)/3, T(2)/9\} \). This implies that \( c3^n \leq T(n) \leq c'3^n \) for \( n = 1, 2 \). Suppose that \( c3^k \leq T(k) \leq c'3^k \) for all \( k, 1 \leq k < n \). Then,
\[
2c3^{n-1} + 3c3^{n-2} \leq T(n) \leq 2c3^{n-1} + 3c3^{n-2}.
\]
Simplifying, we get \( c3^n \leq T(n) \leq c'3^n \).

(d) Using the Iteration Method we get
\[
T(n) = \sum_{i=1}^{k} 1 + \Theta(1),
\]
where \( n^{1/2^k} > 2 \) and \( n^{1/2^{k+1}} \leq 2 \). This implies that \( \log \log n - 1 \leq k \leq \log \log n \). Therefore \( T(n) = \Theta(\log \log n) \).

2. (a) \( T(n) = \sum_{i=1}^{n} \sum_{j=1}^{n} \Theta(1) \). This can be simplified as
\[
T(n) = \sum_{i=1}^{n} \Theta(n - i + 1) = \Theta(n^2 - n(n + 1)/2 + n) = \Theta(n^2/2 + n/2) = \Theta(n^2).
\]

(b) \( T(1) = \Theta(1) \) and \( T(n) = T(n - 1) + \Theta(1) \) for all \( n > 1 \). Using the Iteration Method, we get \( T(n) = \Theta(n) \).

(c) \( T(1) = \Theta(1) \) and \( T(n) = 2T(n/2) + 1 \) for all \( n > 1 \). Using the Master Method, this recurrence solves to \( T(n) = \Theta(n) \).

3. (a) Let a **block** of \( A[1\ldots n] \) be a contiguous subsequence \( A[i], A[i+1], \ldots, A[j] \), for any \( 1 \leq i \leq j \leq n \). Define the **weight** of a block to be the sum of the elements in the block. **Mystery** returns the the weight of the heaviest block in \( A \).
(b) $T(n) = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=i}^{j} \Theta(1)$. This can be simplified as follows:

$$T(n) = \sum_{i=1}^{n} \sum_{j=i}^{n} \Theta(j - i + 1)$$

$$= \sum_{i=1}^{n} \left( 1 - i + n - i(1 - i + n) - \frac{1}{2}(-1 + i - n)(i + n) \right)$$

$$= \frac{1}{2} \left( 2i^2 + 3n^2 + n^3 - i(3 + 2n) \right)$$

$$= \frac{1}{2} \left( 2n + 3n^2 + n^3 + \frac{1}{6}n(1 + n)(1 + 2n) - \frac{1}{2n}(1 + n)(3 + 2n) \right)$$

$$= \frac{n^3}{6} + \frac{n^2}{2} + \frac{n}{3}$$

$$= \Theta(n^3).$$

(c) The idea for a $\Theta(n)$ algorithm is this. For the subarray $A[1...i]$ maintain two pieces of information called $\text{maxSum}[i]$ and $\text{rightMaxSum}[i]$. $\text{maxSum}[i]$ equals the weight of the heaviest block in $A[1...i]$ while $\text{rightMaxSum}[i]$ equals the weight of the heaviest block in $A[1...i]$ that contains $A[i]$. Given this information, the two new pieces of information $\text{maxSum}[i+1]$ and $\text{rightMaxSum}[i+1]$ corresponding to the subarray $A[1...i+1]$ can be computed in $\Theta(1)$ time as follows.

if ($A[i+1] + \text{rightMaxSum}[i] \leq A[i+1]$) then
  $\text{rightMaxSum}[i+1] \leftarrow A[i+1]$
else
  $\text{rightMaxSum}[i+1] \leftarrow \text{rightMaxSum}[i] + A[i+1]$

if ($\text{rightMaxSum}[i+1] \geq \text{maxSum}[i]$) then
  $\text{maxSum}[i+1] \leftarrow \text{rightMaxSum}[i+1]$
else
  $\text{maxSum}[i+1] \leftarrow \text{maxSum}[i]$

Note that $\text{maxSum}[1] = \text{rightMaxSum} = A[1]$ and the answer we want is $\text{maxSum}[n]$.

4. (a) Let $C_i$ denote the outcome of the $i$th coin toss. Probability that Alice wins is

$$\text{Prob}[C_1 = H] + \text{Prob}[C_1 = T \land C_2 = H \lor C_3 = H] = \frac{1}{2} + \frac{1}{8} = \frac{5}{8}.$$  

(b) Let $H_A$, $T_A$, $H_B$, and $T_B$ denote the number of heads and tails obtained by Alice and Bob respectively. There are two possibilities for the relative sizes of $H_A$ and $H_B$: (i) $H_B > H_A$ (ii) $H_B \leq H_A$. Note that $H_B \leq H_A$ is equivalent to the possibility $T_B > T_A$. The two possibilities $H_B > H_A$ and $T_B > T_A$ are disjoint, cover all possibilities, and are equally likely due to symmetry. Therefore, $\text{Prob}[H_A > H_B] = 1/2$.

(c) Let $F$ and $B$ denote the events that the coin is fair and biased respectively. Let $T_1$ and $T_2$ denote the two coin tosses. Then

$$\text{Prob}[F \mid T_1 = H \land T_2 = T] = \frac{\text{Prob}[T_1 = H \land T_2 = T \mid F] \cdot \text{Prob}[F]}{\text{Prob}[T_1 = H \land T_2 = T]}$$

$$= \frac{1/2 \cdot 1/2 \cdot 1/2}{1/2 \cdot 1/2 \cdot 1/2}$$

$$= \frac{1}{4} \cdot \frac{1}{2} \cdot \text{Prob}[T_1 = H \land T_2 = T]$$

$$= 2\text{Prob}[T_1 = H \land T_2 = T]$$

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Similarly, we get

\[ \Pr[B \mid T_1 = H \land T_2 = T] = p(1-p) \cdot \frac{1}{2 \Pr[T_1 = H \land T_2 = T]} \]

For \( p \neq 0.5 \), \( p(1-p) < 0.25 \) and hence it is more likely that the coin is fair.