CS:3330 Homework 1, Spring 2017

Due in class on Thu, Jan 26

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1. Decide whether you think the following statement is true or false. If it is true, give a short explanation. If it is false, give a counterexample.

   > True or false? In every instance of the Stable Matching Problem, there is a stable matching containing a pair (m, w) such that m is ranked first on the preference list of w and w is ranked first on the preference list of m.

   (This problem is from the Chapter 1 exercises in the Kleinberg-Tardos textbook.)

2. Decide whether you think the following statement is true or false. If it is true, give a short explanation. If it is false, give a counterexample.

   > True or false? Consider an instance of the Stable Matching Problem in which there exists a man m and a woman w such that m is ranked first on the preference list of w and w is ranked first on the preference list of m. Then in every stable matching S for this instance, the pair (m, w) belongs to S.

   (This problem is from the Chapter 1 exercises in the Kleinberg-Tardos textbook.)

3. Show the execution of the Gale-Shapley algorithm on an input with 3 men and 3 women having the following preferences:

   $$m_1 : w_2 > w_1 > w_3 \qquad m_2 : w_2 > w_1 > w_3 \qquad m_3 : w_1 > w_2 > w_3$$

   $$w_1 : m_2 > m_1 > m_3 \qquad w_2 : m_3 > m_1 > m_2 \qquad w_3 : m_3 > m_2 > m_1$$

   So man m_1 prefers woman w_2 the most, w_1 next, and w_3 the least. Let us suppose that whenever your algorithm has choice of "free" men to pick for making a proposal, it picks the man with lowest index.

4. Consider the *stable roommate* problem in which we are given n individuals (for even n) and each individual has preferences over the remaining $n - 1$ individuals. (So in this problem, the individuals are not divided into two groups, as in the *stable marriage* problem.) Unlike the stable marriage problem, there are instances of the stable roommate problem for which no solution exists. Consider the following example with 4 individuals A, B, C, and D with the following preferences:

   $$A : B > C > D, \qquad B : C > A > D, \qquad C : A > B > D, \qquad D : A > B > C.$$

   Prove that for this input, there is no solution to the stable roommate problem.

5. Instead of men and women, let us consider the stable marriage problem with hospitals and residents. Each hospital has a preference ordering over the set of residents and each resident has a preference ordering over the set of hospitals. Now suppose that each hospital has a *quota* of residents that it can accept. Just to keep things simple, suppose that this quota is the same for all hospitals and it is a small positive number, say 2. Thus each hospital is interested in being matched with 2 residents. Finally, again to keep things simple. let us assume that there are n hospitals and $2n$ residents. This will ensure that all hospitals' quotas are met.

   Let H be the set of n hospitals and R be the set of $2n$ residents. A set $S \subseteq H \times R \times R$ of triples is a *perfect matching* if each hospital appears exactly once in some triple in S and similarly each resident appears exactly once in some triple in S. An *instability* with respect to S is a pair $(h, r) \in H \times R$ such that hospital h prefers resident r to at least one of the two residents it has been assigned in S *and* r prefers h to the hospital she has been matched with in S. A set $S \subset H \times R \times R$ is a *stable perfect matching* if it is a perfect matching that has no instabilities.

Describe an algorithm that takes as input the sets H, R, the preference orderings of the hospitals in H, the preference orderings of the residents in R, and produces a stable perfect matching.

Note: You do not have to provide a proof of correctness of your algorithm – just a clear description of an algorithm, using the style we used in class to describe the Gale-Shapley algorithm.

6. Problem 7 (Page 26) from the exercises at the end of Chapter 1 from the Kleinberg-Tardos textbook.
