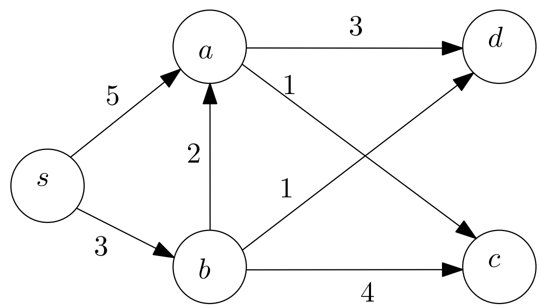
CS:3330 Final Exam, Fall 2015

Monday, Dec 14 2015, 7:30 am to 9:30 am

1. Shortest paths are not always unique: sometimes there are two or more different paths between two vertices, with the minimum possible length. Now consider the following problem.

   Input: A directed, edge-weighted graph $G = (V, E)$, a source vertex $s \in V$
   Output: A boolean array `usp[·]` such that for each vertex $u \in V$, the entry $\texttt{usp}[u]$ is True if and only if there is a *unique* shortest path from s to u.
   Note: $\texttt{usp}[s]$ = True.

   (a) For the following directed, edge-weighted graph and given source vertex s, write down the values in the array `usp[·]`.



   (b) Given a directed graph with n vertices and m edges, describe an algorithm that solves the above problem in $O((m + n) \log n)$ time.
   Hint: Your starting point should be Dijkstra's shortest path algorithm.
   Note: Use the top of the next page for your answer.

2. Consider a data structure that maintain a *binary counter* of unspecified length and supports two operations: (i) `increment`, which increments the counter's value by 1 and (ii) `reset`, that resets the counter's value to 0.

   A simple way to implement a binary counter is to allocate a very large array, say of length M, of bits when the data structure is initialized and set all these bits to 0. Then, the `increment` operation can be implemented as follows:

   function $\texttt{increment}(\texttt{B})$
   $\quad \texttt{i} \leftarrow 0$
   $\quad$ **while** $(\texttt{B}[\texttt{i}] = 1)$ **do**
   $\quad\quad \texttt{B}[\texttt{i}] \leftarrow 0$
   $\quad\quad \texttt{i} \leftarrow \texttt{i} + 1$
   $\quad \texttt{B}[\texttt{i}] \leftarrow 1$

   Notice that this implementation of `increment` does not check for an overflow; it just assumes that M is going to be large enough that checking for overflow is unnnecessary. In your thinking about this problem, do not worry about the possibility of an overflow.

   Example: Suppose that after initializing the binary counter data structure, we perform five `increment` operations. Thus the current value of the counter is 5, which is represented by $\texttt{B}[0] = 1$, $\texttt{B}[1] = 0$, $\texttt{B}[2] = 1$, $\texttt{B}[i] = 0$ for all $i > 2$. Then calling `increment` once more changes $\texttt{B}[0]$ to 0 and $\texttt{B}[1]$ to 1, leaving all other bits unchanged.

   (a) Suppose that the binary counter is initialized as described above. Now consider a sequence of n operations, some of which are `increment` operations and some of which are `reset` operations. What is the worst case running time of any one of these operations, as a function of n?

(b) Argue that the amortized running time of these operations is $O(1)$.
Hint: Every increment is a sequence of assignments that turn a bunch of bits from 1 to 0 followed by one assignment that turns a bit from 0 to 1. Now notice that every assignment that turns a bit from 1 to 0 can be "charged" to a previous `increment` operation that turned that bit from 0 to 1.

3. The following statements may or may not be True. In each case, determine if the statement is True or False. If you claim that the statement is True, provide a proof. Otherwise, provide a counterexample.

   (a) Let $G = (V, E)$ be an undirected, edge-weighted graph and let T be a minimum spanning tree (MST) of G. Now let us increase the weight of every edge in G by 1. T is still an MST of the graph with increased edge-weights.

(b) Let $G = (V, E)$ be a directed, edge-weighted graph and let P be a shortest path in G from a vertex $s \in V$ to a vertex $t \in V$. Now let us increase the weight of every edge in G by 1. P is still a shortest path from s to t in the graph with increased edge-weights.

(c) Suppose a graph G with n vertices has more than $n - 1$ edges and there is a unique heaviest edge. Then this edge cannot be part of any minimum spanning tree of G.

(d) Suppose that G is an undirected, edge-weighted graph in which all edge-weights are distinct and positive. Consider a vertex $s \in V$. It is possible for the tree of shortest paths from s (to all vertices in G) to not share even a single edge with the minimum spanning tree of G.

4. Consider the following recursive function that takes as arguments an array `L` and two non-negative integers `first` and `last`, that serve as indices into `L`. Therefore, if `L` has length n, then `first` and `last` are guaranteed to be in the range 0 through $n - 1$.

```
function strangeSum(L, first, last)
    if (last < first) then
        return 0
    if (last = first) then
        return L[first]
    if (last = first + 1) then
        return L[first] + L[first+1]
    else
        m ← last - first + 1
        leftSum ← strangeSum(L, first, first + m/2 - 1)
        midSum ← strangeSum(L, first + m/4, first + 3 * m/4 - 1)
        rightSum ← strangeSum(L, first + m/2, last)
        return leftSum + midSum + rightSum
```

(a) What is the value returned by the function call `strangeSum(L, 0, 3)` where `L` is the array `[1, 4, 2, 3]`.

(b) Write a recurrence relation describing the running time the function call `strangeSum(L, 0, n-1)` on an array `L` of length `n`.

(c) Solve the recurrence in (b) to obtain the running time of the function call `strangeSum(L, 0, n-1)`, in terms of n, the length of the given array `L`.

5. Here are some problems on NP-completeness and intractability.

   (a) The decision version of the Interval Scheduling problem is the following.

   Interval Sceduling Decision (ISD)
   Input: A set I of intervals, a positive integer k.
   Output: Is there is subset $I' \subseteq I$ of pairwise non-overlapping intervals of size at least k?

   The decision version of the Maximum Independent Set problem is the following.

   Maximum Independent Set Decision (MISD)
   Input: A set $G = (V, E)$, a positive integer k.
   Output: Is there is an independent set $V' \subseteq V$ of size at least k?

   Your task is to prove that ISD $\leq_P$ MISD.

   (b) OK, proving that ISD $\leq_P$ MISD may not have been that difficult. But, what about showing that MISD $\leq_P$ ISD? I want you to either prove that MISD $\leq_P$ ISD or argue that it is unlikely for there to be a polynomial-time reduction from MISD to ISD.

(c) For a graph $G = (V, E)$, a *clique* is a set $C \subseteq V$ of vertices such that every pair of vertices in C is connected by an edge. Now consider the following decision problem:

MAXIMUM CLIQUE DECISION (MCD)
Input: A set $G = (V, E)$, a positive integer k.
Output: Is there is an clique $C \subseteq V$ of size at least k?

Using the fact that MISD is NP-complete, I want you to show that MCD is NP-complete. Recall that there are two main steps in showing this: (i) Show that MCD $\in NP$ and (ii) MISD $\leq_P$ MCD.