# 22C:31 Exam 2

## Due via ICON dropbox by 5:30 pm, Friday, April 2nd

---

You should feel free to use your textbook and notes. However, you should not seek help on the internet or from any other person. By turning in your exam, you are confirming that you've completed your exam within these constraints. You may send me e-mail seeking clarifications; these I can provide, but I will not provide any hints. The exam is worth 150 points. This translates to 15% of your grade.

1. [**50 points**] Solve Problem 1 in Chapter 5 in the textbook. This also happens to be a problem in HW 4.
   **Hint:** It is possible to make two queries, one to each of the databases, and use the answers to reduce the problem to half its original size.

2. [**50 points**] Consider the recurrence

   $$T(n) = qT\left(\frac{n}{2}\right) + n,$$

   where $q$ is a positive integer. You solved a recurrence such as this for $q = 4$ in Homework 3, Problem 3(a). From lectures and from Chapter 5 in the textbook you know that this recurrence has the following solution:

   $$T(n) = \begin{cases} O(n) & \text{if } q = 1; \\ O(n \log n) & \text{if } q = 2; \\ O\left(n^{\log_2 q}\right) & \text{if } q > 2. \end{cases}$$

   The value $q = 2$ seems to be a sort of "threshold" for this recurrence; $T(n)$ behaves in a one way when $q < 2$, in a different way when $q = 2$, and in yet another way when $q > 2$.

   Extend this result to *solve the recurrence*

   $$T(n) = qT\left(\frac{n}{2}\right) + n^d,$$

   where $q$ and $d$ are both positive integers. In Homework 4, you have encountered recurrence relations such as these for $q = 7, 8$ and $d = 2$. Your solution should look similar to the solution for the recurrence $T(n) = qT\left(\frac{n}{2}\right) + n$. More specifically, you will find that there is a "threshold" (that depends on $d$) such that for values of $q$ less than this threshold, $T(n)$ behaves in one way, for $q$ equal to this threshold, $T(n)$ will behave in a different way, and finally for values of $q$ above this threshold $T(n)$ will behave in yet another way.

   Present your answer neatly, first writing down the solution to the recurrence and then showing the calculations that "prove" your solution.

   **Hint:** It will be helpful to remind yourself that the geometric series $\sum_{i=0}^{T} a^i$ is bounded above by $1/(1-a)$ when $|a| < 1$; $(T+1)$ when $a = 1$; and $(a^{T+1} - 1)/(a-1)$ when $a > 1$.

3. In many network design applications, we want to find a spanning tree that minimizes, not the sum of the costs of the edges, but the cost of the costliest edge in the spanning tree. More precisely, define the *minimum bottleneck spanning tree* problem as follows. Let $G = (V, E)$ be a connected graph in which each edge $e \in E$ has a positive cost $c(e)$. We want to find a spanning tree $T = (V, E')$ of $G$ such that $\max_{e \in E'} c(e)$ is minimum (over all possible spanning trees of $G$). Such a spanning tree is called a *minimum bottleneck spanning tree* (MBST, in short).

(a) **[25 points]** Is every MBST also a minimum spanning tree? Answer **True** or **False**. If your answer is **True**, present a proof. If your answer is **False**, present a counterexample.

(b) **[25 points]** Is every minimum spanning tree also an MBST? Answer **True** or **False**. If your answer is **True**, present a proof. If your answer is **False**, present a counterexample.

---