**Algorithmic Version of $L^3$**

BECK 1991

Given the proof of existence of a certain object via $L^3$, is it possible to efficiently construct it?

k-SAT

The problem k-SAT is the following:

*Input:* A set of classes in disjunctive normal form so that each clause has exactly k literals.

*Question:* Is there a satisfying truth assignment to the variables?

k-SAT (1)

*Input:* Same as k-SAT except that each variable occurs in exactly 1 clauses.

*Question:* Is there a satisfying truth assignment to the variables?

Consider k-SAT($2^{k/50}$), e.g. 100-SAT(4)

*Claim:* k-SAT($2^{k/50}$) is always satisfiable.

*Proof:*

To each variable assign at random the value T or F independently and with equal probability. Let C be a clause.

\[ \text{Prob.}[C \text{ is not satisfied}] = 1/2^k. \]

Let $A_C = C$ is not satisfied.

\[ \text{Prob.}[\bigwedge C : A' C ] > 0 \iff \text{there is a truth assignment satisfying all clauses.} \]

What is d for \( \{ A_C \mid C \text{ is a clause} \} \)?

\[ d \leq k \cdot 2^{k/50} \]

so $e.p(d+1) = e.(1/2^k)(k.2^{k/50} + 1) \leq 1$ for all $k \geq 4$.

*Comment:* 2-coloring uniform hypergraphs is another context in which Beck’s algo. is described.

Is there an efficient algo. to find a satisfying truth assignment for k-SAT ($2^{k/50}$) ?

We will describe an algorithm that runs in polytime in m(number of clauses) but not in k.

*Implication:* We have a poly-time algo. for $k=O(1)$.

Consider for example: MAX-3SAT(6), which is quite hard to approximate.

*Algorithm:*

Stage 1:

- Order the variables in some arbitrary order.
- Process the variables in this order, assigning to each var. the value T or F with equal prob., except if the var. belongs to a dangerous clause.

A clause C is dangerous if

i) C has $k/2$ literals that have been assigned a value.

ii) C is not satisfied.

At the end of stage 1 we have some satisfied clauses and some surviving clauses. A surviving clause is a clause in which no literal has been assigned the value T.

*Remark:*

A clause may be surviving because all its unassigned variables participate in dangerous clauses.
Let $G$ be the graph whose vertex set is the set of clauses and whose edge set

$$\{ \{ C_1, C_2 \} \mid C_1 \text{ and } C_2 \text{ share a variable} \}$$

Note that degree $G(C) \leq k \cdot 2^{k/50} = d$.

Let $H$ = subgraph of $G$ induced by surviving clauses and unassigned variables.

We can show that with high prob. $(1 - O(1))$ every connected component of $H$ is $O(\log m)$ in size.

Lemma:

With prob. $(1 - O(1))$ every connected component of $H$ is of size $\leq Z \cdot \log(m)$ for some fixed constant $Z$.

Proof:

Let $C_1, C_2, \ldots, C_r$ be a collection of clauses such that $\text{dist}_G(C_i, C_j) \geq 4$ for any $i \neq j$.

We will now discuss the prob. that $C_1, C_2, \ldots, C_r$ all survive stage 1. $C_i$ survives only because some clause in $\{C_i\} \cup N(C_i)$ is dangerous.

Prob. $[a$ clause becomes dangerous in stage 1] $= 1/2^{k/2}$.

To each $C_i$ one can associate a clause $D_i \in \{C_i\} \cup N(C_i)$ that becomes dangerous in stage 1.

Prob. $[a$ all of $D_i s$ are dangerous $] = (1/2^{k/2})^r$. 

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