22C:253 Lecture 7

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1 A Quick Recap.

Last time, we looked at situations in which LP has integral solutions. Consider an LP:

$$\min\{c^T x | Ax \leq b, x \geq 0\}$$

where $c \in \mathbb{R}^n, b \in \mathbb{R}^m, A \in \mathbb{R}^{m \times n}$

Theorem 1 *All vertices of the feasibility polytope are integral if A is totally unimodular* (TUM) *and b is integral.*

Proof: Any vertex v of the polytope is the intersection of atleast n-dimensional hyperplanes described by the L.P. constraints. In other words, there exists an $n \times n$ matrix A_s and $b_s \in \mathbb{R}^n$ such that v is described by:

$$A_s \cdot x = b_s$$

Note that some of these equations come from the non–negativity constraints. (Basically, we take $m+n$ of the inequalities and then we turn them into equalities.)

$$\begin{aligned} x &= A_s^{-1} \cdot b_s \\ &= \frac{1}{\det(A_s)} \cdot \operatorname{adj}(A_s) \cdot b_s \end{aligned}$$

Recall,

$$\det(A) = \sum_{j=1}^{n} a_{ij} \cdot A_{ij}, \text{ for any } i$$

where A_{ij} is a cofactor of A and A_{ij} is defined as follows:

$$A_{ij} = (-1)^{i+j} \cdot \det(M_{ij}$$

where M_{ij} is called the "minor".

$$\operatorname{adj}(A) = \begin{pmatrix} A_{11} & \dots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \dots & A_{nn} \end{pmatrix}$$

If A is TUM, then $\det(A_s) \in \{\pm 1\}$.
Also, $\operatorname{adj}(A)$ is a matrix where $A_{ij} = \{\pm 1\}$.
It follows that if b is integral, then $\frac{1}{\det(A_s)} \cdot \operatorname{adj}(A_s) \cdot b_s$ is an integral vector. □

2 Application of the Theorem

2.1 Maximum Matching Problem in a Bipartite Graph

Let $G = (V, E)$ be a bipartite graph. Express the maximum matching problem on G as an I.P. Let $x_e \in \{0, 1\}$ be an indicator variable denoting the presense of edge e in the solution:

$$\max \sum_{e \in E} x_e$$

such that,

$$\sum_{e \text{ isincident on } v} x_e \leq 1, \text{ for } v \in V$$

and,

$$x_e \in \{0, 1\}, \text{ for } e \in E$$

The corresponding relaxation replaces $x_e \in \{0, 1\}$ by $x_e \geq 0$.

Claim 1 *This relaxation always has an integral solution.*

Also, consider the L.P. relaxation of the I.P. for the Vertex Cover (VC) problem.

$$\min \sum_{v \in V} c_v \cdot x_v$$

such that,

$$x_u + x_v \geq 1, \text{ for each edge } e = \{u, v\}$$

$$x_v \geq 0, \text{ for each } v \in V$$

What is the matrix corresponding to contraints other than the non-negativity constraints?
Each row of the matrix corresponds to vertices in G and each column corresponds to edges in G. An element $A_{ve} \in \{0, 1\}$ represents whether an edge e is incident upon vertex v in G. This matrix is also known as an *incidence matrix.*

Claim 2 *The incidence matrix of a bipartite graph is* TUM.

2.2 Vertex Cover Problem

Likewise, consider an I.P. for the vertex cover problem of a bipartite graph. What does the matrix look like?
Each row of the matrix corresponds to edges in G and each column represents vertices of G. An element in the matrix tells us if an edge e is incident upon vertex v. This matrix is just the transpose of the incident matrix.

Theorem 2 *Let A be a matrix with entries in the set $\{-1, 0, +1\}$, such that, each column has atmost two non–zero entries. Suppose the rows of A are partitioned in two sets, namely, I_1, I_2, such that,*

1. *If a column contains two non–zero entries of the same sign then they appear in differnt partitions.*
2. *If a column contains two non–zero entries of different signs then they appear in the same partition.*

Subject to the above conditions, A is TUM.

Proof: By induction on the size of sub–matrices.
Base Case:
Claim trivially true for a 1 matrix.
Inductive Case:
Consider a $k \times k$ submatrix C. There are two cases:

- If C has a column with all zeros then $\det(C) = 0 \Rightarrow C$ is singular.
- If C has a column with exactly one non zero entry. Let this entry be in position (i, j). Then,

$$\det(C) = (-1)^{i+j} \cdot M_{ij}$$

  where M_{ij} is obtained by deleting row i and column j from C. An immediate implication then is that $\det(C) \in \{0, \pm 1\}$.
- All columns of C has two non-zero elements. This implies that the sum of all rows of C in I_1 = sum of all rows of C in I_2. Which, in turn, implies that the rows of C are **not** linearly independent, which implies that $\det(C) = 0$.

□

3 Implication of the Theorem

A direct implication of the above theorem is that the incidence of a bipartite graph is TUM.

Corollary 1 *The incidence matrix of any directed graph is* TUM.

Proof: Denote each incoming edge with a $+1$ and outgoing edge with a -1. After that, the application of the above theorem is trivial. □

4 Half–Integrality of the Vertex Cover Problem

In this section we present a remarkable result due to G. Nemhauser and L. Trotter.

Definition 1 *A point $x \in \mathbb{R}^n$ is* **half–integral** *if $x_i \in \{0, \frac{1}{2}, 1\}, \forall i \in \{1, 2, \ldots, n\}$*

Theorem 3 *Any vertex of the feasibility polytope of the* VC *problem is half–integral.*

Proof: Assume the contrary. Hence there exists a vertex of the feasibility polytope of VC that is not half–integral. Let $x \in \mathbb{R}^n$ be that vertex. Let

$$V_+ = \{i | \frac{1}{2} < x_i < 1\}, \text{ and } V_- = \{i | 0 < x_i < \frac{1}{2}\}$$

We are assuming that $V_+ \cup V_- \neq \emptyset$ and $x = (x_1, \ldots, x_n)$.
For $\epsilon > 0$, define two new points in $\mathbb{R}^n$ as follows:

$$y_i = \begin{cases} x_i - \epsilon & \text{if } i \in V_+ \\ x_i + \epsilon & \text{if } i \in V_- \\ x_i & \text{otherwise} \end{cases} \quad , \quad z_i = \begin{cases} x_i + \epsilon & \text{if } i \in V_+ \\ x_i - \epsilon & \text{if } i \in V_- \\ x_i & \text{otherwise} \end{cases}$$

Note:

$$y \neq x, z \neq x \tag{1}$$

$$x = \frac{1}{2}(y + z) \tag{2}$$

According to Claim 3 (proved subsequently), ϵ can be made small enough so that both y and z are feasible.
This implies that x is the convex combination of two points in the feasibility polytope.
The above in turn implies that x is **not** a vertex of the feasibility polytope. A contradiction.
So, x is **half–integral**. □

Claim 3 *Given x, y, z and ϵ in the above ϵ can be made small enough so that both y and z are feasible.*

Proof: Since x is a vertex of the feasibility polytope, x is feasible. This implies that,

$$x_i + x_j \geq 1, \text{ for all edges } \{i, j\}.$$

Consider all edges $\{i, j\}$, such that,

$$x_i + x_j > 1$$

and pick $\epsilon > 0$ small enough so that all such edges

$$y_i + y_j \geq 1 \;\&\; z_i + z_j \geq 1$$

Now, we consider constraints that hold tightly for x. In other words,

$$x_i + x_j = 1$$

Look at such an edge $\{i, j\}$. The only possible cases are:

$x_i = x_j = \frac{1}{2} \Rightarrow i \notin V_+ \cup V_-, j \in V_+ \cup V_- \Rightarrow y_i = z_i = x_i, y_j = z_j = x_j$
$x_i > \frac{1}{2}; x_j < \frac{1}{2} \Rightarrow i \in V_+, j \in V_- \Rightarrow y_i = x_i - \epsilon, y_j = x_j + \epsilon \Rightarrow y_i + y_j = x_i + x_j = 1;$ Similarly, $z_i + z_j = 1$
Symmetric case to the above

□