22C:253 Lecture

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9th October 2002

Capacitated Vertex Cover

A factor-3 approximation algorithm for capacitated vertex cover
To prove this we use a very nice technique called Dependent Rounding

Dependent Rounding :
Given a bipartite graph $G = (A, B, E)$ such that each edge $\{i, j\} \in E$ has an associated real number $x_{ij} \in [0, 1]$

Goal:
To devise an efficient procedure that rounds each x_{ij} probabilistically to $X_{ij} \in \{0, 1\}$ such that the following properties are satisfied

(P1) Prob$[X_{ij} = 1] = x_{ij}$

(P2) Let

$$d_i = \sum_{j:\{i,j\}\in E} x_{ij}$$

this is called the fractional degree of i
Let

$$D_i = \sum_{j:\{i,j\}\in E} X_{ij}$$

this is called the integral degree of i
Key Property:

Prob$[D_i \in \{\lfloor d_i \rfloor, \lceil d_i \rceil\}] = 1$
If d_i is integral then $D_i = d_i$ with certainty

(P3)Negative correlation property

Cosider the following:
Assign $X_{ij} = 1$ with probabiliy x_{ij} and $X_{ij} = 0$ with probabiliy $1 - x_{ij}$ independently
Clearly (P1) is satisfied , however (P2) is not

Why is dependence useful:
Cosider randomized rounding for set cover, we are not guaranteed to get a feasible solution, that is the motivation for the property (P2) in which we do dependent rounding to get a feasible solution (P2) works only for bipartite graphs.

The Dependent Rounding scheme:

Let $x_{ij} \in [0,1]$ be a variable associated with edge $\{i,j\} \in E$
Initially $y_{ij} = x_{ij}$ for each $(i,j) \in E$. We probabilistically modify y_{ij} in atmost $|E|$ steps such that $y_{ij} \in \{0,1\}$ at the end.
Then we set $X_{ij} = y_{ij}$ for all $(i,j) \in E$
Our iterations will satisfy the following two invariants:

(I1) For all $(i,j) \in E, y_{ij} \in [0,1]$.

(I2) Call $(i,j) \in E$ rounded if $y_{ij} \in \{0,1\}$, and floating if $y_{ij} \in (0,1)$.Once an edge gets rounded, y_{ij} never changes.

An iteration proceeds as follows. Let $F \subseteq E$ be the current set of floating edges. If $F = \phi$, we are done.Otherwise, find(in linear time) a simple cycle or maximal path P in subgraph (A, B, F), and partition the edge set of P into two matchings M_1 and M_2.Note that such partitions exists since (A,B,E) is a bipartite graph.
If P is a cycle then it is an even cycle

Define
$\alpha =$ maximum mass we can transfer from M_1 to M_2.

$$\alpha = min\{x_{ij} | \{i,j\} \in M_1\} \cup \{1 - x_{ij} | \{i,j\} \in M_2\}$$

$\beta =$ maximum mass we can transfer from M_2 to M_1.

$$\beta = min\{x_{ij} | \{i,j\} \in M_2\} \cup \{1 - x_{ij} | \{i,j\} \in M_1\}$$

We can transfer α from M_1 to M_2 with probability $\frac{\alpha}{\alpha+\beta}$ or
β from M_2 to M_1 with probability $\frac{\beta}{\alpha+\beta}$

Capacitated vertex cover:

$$min \sum_{v \in V} w_v x_v$$

such that :

$$y_{e,v} + y_{e,u} \geq 1 \quad \forall e = (u,v) \in E$$

$$\sum_{e:v\in e} y_{e,v} \leq k_v x_v \quad \forall v \in V$$

$$x_v \geq y_{e,v} \quad \forall v, \forall e : v \in e$$

$$x_v \geq 0 \ \forall v \in V$$

$$y_{e,v} \geq 0 \quad \forall e \in E, v \in e$$

Threshold and Round:

Step1: Solve the LP-Relaxation , Let solution be (X, Y)
Step2(Threshold): For each edge $e = \{u, v\} \in E$, if $y_{e,u} \geq \frac{2}{3}$ set $y^*_{e,u}$ to 1 else if $y_{e,v} \geq \frac{2}{3}$ set $y^*_{e,v}$ to 1.
Step3(Rpunding Step):
Let E' be the set of edges e for which $y_{e,u} \leq \frac{2}{3}$ and $y_{e,v} \leq \frac{2}{3}$
Create a bipartite graph $H = (E', V, E'')$ where V = vertices of the original given graph.
E' connects each edge E in E' to its end points in V.Constraint (P2) is satisfied

Run dependent rounding on H to get $y^*_{e,u}$ valus for the remaining edges $e \in \{u, v\}$
We finally set

$$x^*_v = \lceil \frac{\sum_{e:v\in e} y^*_{e,v}}{k_v} \rceil$$

$\forall v \in V$

Claim:

$$E[X^*_v] \leq 3x_v \ \forall v \in V$$

$$E[\sum_{v\in V} w_v x^*_v] \leq 3 \sum_{v\in V} w_v x_v \leq 3OPT$$

Proof:

Let

$$y_v^* = \sum_{e:v\in e} y_{e,v}^*$$

$$x_v^* = \lceil \frac{y_v^*}{k_v} \rceil$$

$$y_v^* = d_v^* + r_v^*$$

where d_v^* is the number of edges assigned to V in the threshold step and r_v^* is the number of edges assigned to V in dependent rounding step

r_v^* is the integral degree of V

$$r_v^* \in \{\lfloor r_v \rfloor, \lceil r_v \rceil\}$$

where r_v is the fractional degre of v.
so,

$$r_v = \sum_{e:v\in e, e\in E'} y_{e,v}$$

Suppose

$$r_v = a_v + f_v$$

where a_v is the integral part and f_v is the fractional part

$$r_v^* \in \{a_v, a_v + 1\}$$

We can say that,

$$Prob[r_v^* = a_v] = 1 - f_v$$

$$Prob[r_v^* = a_v + 1] = f_v$$

because that is how we performed the dependent rounding

$$E[x_v^*] = (1 - f_v)\lceil \frac{d_v^* + a_v}{k_v} \rceil + (f_v)\lceil \frac{d_v^* + a_v + 1}{k_v} \rceil$$

Proof:
Two cases:
CASE1:

$$d_v^* + a_v < k_v$$

$$d_v^* + a_v + 1 < k_v$$

implies that $E[X_v^*] \leq 1$
We only have to show that $x_v \geq \frac{1}{3}$

If $d_v \geq 1$, then $y_{e,v} \geq \frac{2}{3}$ for some edge e incident on V.
so, $y_{e,v} \geq \frac{2}{3}$ and $x_v \geq \frac{2}{3}$
If $d_v = 0$ and there is some edge in $e \in E'$ incident on V then

$$y_{e,v} > \frac{1}{3}\ ,\ x_v > \frac{1}{3}$$

If $d_v = 0$ and there is some edge $e \in E'$ incident on V then all the edges were already assigned to other end points which implies that $x_v = 0$

CASE2: $d_v^* + a_v \geq k_v$

$$E[X_v^*] \leq (1 - f_v)(\frac{d_v^* + a_v + k_v}{k_v}) + f_v(\frac{d_v^* + a_v + 1 + k_v}{k_v})$$

$$\leq \frac{d_v^* + a_v + k_v}{k_v} + \frac{f_v}{k_v} \leq 2(\frac{d_v^* + k_v}{a_v}) + \frac{f_v}{k_v} \leq \frac{2(d_v^* + a_v + f_v)}{k_v}$$

Claim:

It can be written as

$$3[\frac{\frac{2}{3}(d_v^* + a_v + f_v)}{k_v}] \leq 3x_v$$

Proof
We know that the sum $a_v + f_v$ is the number of edges we have added to the vertex V in the

dependent rounding phase d_v^* is the number of edges we have added during the threshold phase, where we have set the threshold as $\frac{2}{3}$ We can say that we have atlest a value greater than equal to $\frac{2}{3}$ for the x_v's for the vertices
Hence, we have that

$$3[\frac{\frac{2}{3}(d_v^* + a_v + f_v)}{k_v}] \leq 3x_v \leq 3OPT$$