1 Watts-Strogatz Model

For a positive integer $n$ and an even integer $k$, let $C(n,k)$ denote the graph with vertex set \{0, 1, 2, ..., $n - 1$\} and edge set \{{}i, j\}, $0 \leq i, j \leq n - 1, \mid i - j \mid \leq k/2\}.

Figure 1: $C(8,4)$, newly rewired edges are excluded from future rewire.

The Watts-Strogatz graph$[2]$, denoted $WS(n,k,p)$ is obtained from $C(n,k)$ by replacing each edge in $C(n,k)$ with probability $p$ by a randomly chosen edge.
Figure 2: $C(P)$ represents the expected clustering coefficient of $WS(n, k, p)$. $L(P)$ represents the expected average path length of $WS(n, k, p)$. It is shown that every 20 vertices receives a rewire of edges.

2 Discussion

Proximity. The base graph $C(n, k)$ starts by connecting vertices that are close by. For example, $Grid(n, r)$: vertex set $\{0, 1, \ldots, n - 1\} \times \{0, 1, \ldots, n - 1\}$ and edge set $\{(i_1, j_1), (i_2, j_2)\}$, where $|i_1 - i_2| + |j_1 - j_2| \leq r$. An example of $Grid(4, 2)$ is shown in Fig. 3.
It can be more abstract. Let $M = (V, d)$ be a metric space. Consider the graph with vertex set $V$ and edge set: $\{u, v : d(u, v) \leq r\}$, where $r$ is some parameter.

Randomness is added in a variety of ways to achieve the same effect. Alternative Approach: start with $C(n, k)$. To each vertex $u$, add an edge $\{u, v\}$ with $v$ chosen randomly.

Result. This result made a lot of sense to sociologists because they believed in two types of edges:

1. edges induced by homophily $\implies$ base graph edges
2. edges that correspond to weak ties $\implies$ random edges
   
   homophily + weak ties $\implies$ small world property.

Recall.
Figure 4: Alternate edges/connections may dampen the size of the set and may elongate the graph. Alternate edges/connections can also make the lengths to other edges quicker.

Clustering Coefficient↑ ⇒ Average Path Length ↑

**Diameter.** The Diameter of a Cycle Plus a Random Matching[1]. See Fig. 5. for example.

\footnote{Node based definition of clustering coefficient.}
3 Kleinberg’s Question

Watts-Strogatz model[2] is small world. Does it also allow efficient decentralized (local) search?

Example. Consider WS(n, k, p). Let s and r denote sender and receiver. The sender, knowing only r’s label, has a package that needs to be sent to r. Typical Step: node v on receiving the package, either:

1. If v has a neighbor closer to r than itself, v sends the package to the neighbor closest to r.
2. Otherwise, v gives up.

Questions. Suppose we pick s and r randomly and perform graphic greedy routing many times:

- What fraction of these experiments is successful?
- What is the average path length of the successful experiments?

4 Kleinberg’s Model[3]

Let us use K(n, r, q, −α) to denote the graph obtained by starting with Grid(n, r) and adding random edges as follows: To each vertex u, add q random edges {u, v} with v picked out with a probability proportional to d(u, v)−α. If α = 2, then d(u, v)−α = 1/d(u, v)^2. Fig. 6 demonstrates the model.
Figure 6: One hop edges have higher probability to be connected than 2 or more hop edges. If $\alpha = 1$, then the probability distribution is uniform such that no differentiation between near neighbors and far nodes.

Results.

1. For $\alpha = 0$, any decentralized algorithm requires at least $(n^{2/3})$ hops
2. For $\alpha = 2$, geographic greedy routing discovers paths of expected length $O(\log 2n)$
Figure 7: The correlation between $\alpha$ and the expected path length.

References

