1 Game-theoretic Modeling of Traffic Congestion

Consider the following algorithm that converges to an N.E.[1]:

\[ \text{GMTG}(G, (p_1, p_2, \ldots, p_n)) \]

1. start with an arbitrary strategy choice \((p_1, p_2, \ldots, p_n)\).
2. while \((p_1, p_2, \ldots, p_n)\) is not a N.E.:
3. pick \(p_i\) that is not the best response to the remaining choices \((p_1, p_2, \ldots, p_{i-1}, p_{i+1}, \ldots, p_n)\).
4. replace \(p_i\) by a shorter path \(p_i\) (a better response)

**Theorem 1** Every traffic congestion game has a pure strategy N.E.

**Proof:** Define a potential function and show each iteration improves the function in the finite space of pure strategies. For an edge \(e \in E\) across which \(x\) cars are traveling in \((p_1, p_2, \ldots, p_n)\), define

\[ \text{Potential}(e) = T_e(1) + T_e(2) + \ldots + T_e(x) = \sum_{j=1}^{x} T_e(j) \]

\[ \text{Potential}(p_1, p_2, \ldots, p_n) = \sum_{e \in E} \text{Potential}(e) \]

In a typical iteration of the while-loop, let \(e\) be an arbitrary edge on path \(p_i\). Let \(x\) be the number of cars traveling across \(e\) just before this iteration. \(\text{Potential}(e)\) decreases by \(T_e(x)\). That is, potential along \(p_i\) decreases by the sum of travel times along \(p_i\). By the same argument, potential along \(e\) increases by \(T_e(y)\) where \(y\) = volume of traffic along \(e\) after \(i\) joins this edge. In other words, potential along \(p_i\) increases by travel time along \(p_i\). Since travel time along \(p_i\) is strictly smaller than travel time along \(p_i\), the potential strictly falls \(\sum_{j=1}^{x} T_e(j)\) This shows that there always exists a pure strategy N.E. \(\square\)

**Theorem 2** For every traffic congestion game, there is a pure strategy N.E.(not every) whose cost is within 2 times of the cost of the welfare maximizing solution

**Proof:** Let \(Z^*\) be a welfare maximizing strategy choice. In other words, \(Z^*\) minimizes the sum of travel times of all players. Let \(\text{TTT}(Z)\) denote the sum of travel times of all players for a strategy choice \(Z\). \(xT_e(x)\) is the contribution of \(e\) to \(\text{TTT}(Z)\).

\[ \text{Potential}(e) = T_e(1) + T_e(2) + \ldots + T_e(x) \]

\[ \text{TTT}(e) = T_e(x) + T_e(x) + \ldots + T_e(x) = xT_e(x) \]

Suppose \(T_e(y) = a_ey + b_e\), the following figure shows the difference of the areas that represent \(\text{Potential}(e)\) and \(\text{TTT}(e)\).
We have shown $\frac{TTT(e)}{2} \leq Potential(e) \leq TTT(e)$. For any strategy choice $Z: \frac{TTT(Z)}{2} \leq Potential(Z) \leq TTT(Z)$ Suppose we run our algorithm starting from $Z^*$ in the computation, $Z^* \rightarrow Z$ (potential falls, $TTT$ might increase)

$$\Rightarrow \frac{TTT(Z)}{2} \leq Potential(Z^*) \leq TTT(Z^*)$$
$$\Rightarrow TTT(Z)/2 \leq Potential(Z)$$
$$\Rightarrow TTT(Z)/2 \leq TTT(Z^*)$$
$$\Rightarrow TTT(Z) \leq 2TTT(Z^*)$$

$\square$

References