1 Introduction to Game Theoretic Modelling of Traffic Congestion

Example 1: This is the first example of game on graph. This game is played by 4000 drivers. In the graph, A is the source and B is the target. The labels on the edges of the graph represent the delays on the route. The driver can choose any path among the two.

![Diagram](image_url)

Figure 1: Diagram for the Example 1 showing the NE for the example in which 2000 drivers goes on both the paths

The travel time for each driver, when both routes have 2000 drivers will be: \(20 + 45 = 65\) units of time.

Ques) Suppose we use the sum of travel time as the social welfare function than how good is the NE solution, relative to a choice that minimizes total travel time?

Suppose, \(p\) drivers travel on path \(A \rightarrow C \rightarrow B\) and the remaining \(4000-p\) travels on \(A \rightarrow D \rightarrow B\)

\[
\min_{0 \leq p \leq 4000} \left( \frac{p}{100} + 45 \right)p + (4000 - p)(\frac{4000 - p}{100} + 45) \tag{1}
\]

\[
= \min_{0 \leq p \leq 4000} \frac{p^2}{100} + 45p + \frac{(4000 - p)^2}{100} + 4000 \times 45 \tag{2}
\]

\[
= \min_{0 \leq p \leq 4000} p^2 + (4000 - p)^2 \tag{3}
\]
Figure 2: Graph showing that the incentive to deviate minimizes at 2000

NE is equal to welfare maxim choice.

**Example 2:** Figure: 1 refers to the problem in example 2, where there are have 4000 drivers with the choice of three paths.

![Diagram for Example 2 showing the NE extra capacity](image)

Figure 3: Diagram for the Example 2 showing the NE extra capacity

**Ques:** Is the NE from the previous example still a NE?

**Answer:** No, as a driver can do better by switching to $A \rightarrow C \rightarrow D \rightarrow B$ because the travel time on this path is much less than 65.

**Ques:** What is the NE solution?

**Answer:** Consider, $a$ and $b > 0$, a solution in which 'a' drivers take $A \rightarrow C \rightarrow D \rightarrow B$ and 4000 - ($a + b$) take the path $A \rightarrow C \rightarrow D \rightarrow B$.

Travel time on path $A \rightarrow C \rightarrow B$:

$$\frac{4000-b}{100} + 45 = 85 - \frac{b}{100}$$

Similarly, travel time on path $A \rightarrow D \rightarrow B$:

$$85 - \frac{a}{100}$$

and travel time on path $A \rightarrow C \rightarrow D \rightarrow B$:

$$80 - \frac{a}{100} - \frac{b}{100}$$

No, solution in which a driver uses $A \rightarrow C \rightarrow B$ or $A \rightarrow D \rightarrow B$ is NE. So, all 4000 drivers will travel on path, $A \rightarrow C \rightarrow D \rightarrow B$. This is the NE solution.
Observations:

1. Braess Paradox: Adding a capacity to the network led to a NE solution in which everyone is worse off.

2. The solution that maximizes social welfare is strictly better than the unique NE.

\[
\frac{\text{cost of NE}}{\text{Cost of social welfare maximum solution}} = \text{Price of anarchy}
\]

In this example, price of anarchy $\geq \frac{80}{85}$

Results: (Roughgarden and tardos)

1. For any traffic network $(G = (V, E), s,t)$, this game has a pure strategy NE.

2. For any traffic network $(G = (V,E), s,t)$, price of anarchy $\leq 2$. [More complex algorithm bond it by $\frac{4}{3}$]

For linear delay functions, i.e.,

\[ T_e(x) = a_e x + b_e \]

Consider the following “natural” algorithm:

Start with $(p_1, p_2, ..., p_n)$ is the choice of an $st$-path for each of the players $1, 2, ..., n$.

while $(p_1, p_2, ..., p_n)$ is not a NE do

Pick a player $i$ who is not playing her best response and replace $P_i$ by a path $P_i'$ with strictly shorter travel time.

end while

Existence of a pure strategy NE: Generally this type of algorithm is known as Tatomeat, and generally they don’t converge but for the example 2, this algorithm converges when we find a path optimal for one $p$. 