22C:196 Homework 4

Due: Wednesday, 12/18 at final exam

Notes: (a) Computer Science Ph.D. students are required to solve Problems 1, 5, 6, and 7. The rest of the students are required to solve the Problems 1-4. The problem numbers with no additional qualifications refer to problems in the textbook, by Mitzenmacher and Upfal. (b) It is possible that solutions to some of these problems are available to you via other textbooks, on-line lecture notes, etc. If you use any such sources, please acknowledge these in your homework with a complete citation *and* present your solutions in your own words. You will benefit most from the homework, if you sincerely attempt each problem on your own first, before seeking other sources. (c) As mentioned in the syllabus, it is *not* okay to discuss these problems with your classmates. But, you are welcome to come and chat with me about the problems. (d) Students who are not Computer Science Ph.D. students will receive extra credit for submitting correct solutions to any non-empty subset of Problems 5-7.

1. Problem 4.25

2. For this problem suppose that when we say `MAXSAT`, we mean the unweighted version of `MAXSAT`. Let (x^{LP}, y^{LP}) be an optimal solution to the `MAXSAT` LP-relaxation discussed in class. Suppose we produce a solution to `MAXSAT` (i.e., a truth assignment to variables $x_1, x_2, \ldots, x_\ell$) by using the following simple, deterministic rounding procedure: if $x_i^{LP} \geq 1/2$ then $x_i^R := 1$; otherwise $x_i^R := 0$. Do you think this algorithm yields an α-approximation to `MAXSAT`, for some constant α? Justify your answer, either with a proof (or analysis) or with a counterexample.

3. Here is yet another approximation algorithm for (unweighted) `MAXSAT`.

   (a) Suppose the `MAXSAT` instance is such that all the length-1 clauses (also called "unit clauses") only contain variables that are un-negated. Now suppose we set each variable to 1 with probability $p \geq 1/2$ and 0 otherwise. Show that each clause is satisfied with probability at least $\min(p, 1 - p^2)$. Hence show that for a suitable value of p, the expected number of clauses satisfied is at least $(\sqrt{5} - 1)m/2 \approx 0.618m$, where m is the total number of clauses.

   (b) Now consider a formula without any restriction on the unit clauses. Clearly, we cannot hope to satisfy more than 50% of the clauses, since the formula might be $(x) \wedge (\overline{x})$. However, show an algorithm that satisfies at least $c \cdot OPT$ of the clauses, where OPT is the maximum number of clauses that can be satisfied by any assignment and $c = (\sqrt{5} - 1)m/2$.

4. Derandomize the $(1 - \frac{1}{e})$-approximation algorithm for `MAXSAT` using the method of conditional expectations. Carefully describe your deterministic algorithm and argue that (i) it produces an $(1 - \frac{1}{e})$-approximation and (ii) it runs in polynomial time.

5. Let G be a $\sqrt{n}$-regular graph (i.e., a graph in which every vertex has degree $\sqrt{n}$). Suppose that vertices are deleted from G independently with probability $1 - 1/(3n^{1/4})$.

   (a) Use the Lovász Local Lemma to argue that with positive probability, the subset of vertices that survive, form an independent set.

(b) Use the Chernoff bound to show that the probability that fewer than $n^{3/4}/6$ vertices survive is less than $\exp(-n^{3/4}/12)$.

(c) Now consider what happens when you run the random deletion process on an $\sqrt{n}$-regular graph containing no independent set of size more than $\sqrt{n}$. What does this say about the positive probability in part (a)?

(This is Problem 5.8 in "Randomized Algorithms" by Motwani and Raghavan.)

6. Let G be a d-regular graph on n vertices.

   (a) Show that the number of connected subgraphs of G of size r is at most nd^{2r}.

   (b) Suppose that each vertex of G is deleted independently with probability $1 - \frac{1}{2d^2}$. Show that with probability $1 - n^{-\alpha}$, there is no surviving connected component of size exceeding $\log n$, for suitable constant α.

   (This is Problem 5.6 in "Randomized Algorithms" by Motwani and Raghavan.)

7. This problem guides you towards derandomizing the randomized rounding algorithm for set cover. Let X_j be a random variable that indicates whether set S_j has been included in the solution. Let w_j be the weight of S_j. Thus $W := \sum_j w_j X_j$ is the random variable denoting the weight of the solution produced by the algorithm. Now let Z be a random variable such that $Z = 1$ if randomized rounding does not produce a valid set cover, and $Z = 0$ if it does. Then consider applying the method of conditional expectations to the objective function $X + \lambda Z$ for some choice of $\lambda \geq 0$. Show that for the proper choice of λ, the method of conditional expectations applied to the randomized rounding algorithm yields an $O(\ln n)$-approximation algorithm to the set cover problem.
   (This is Problem 5.7 in "The design of Approximation Algorithms" by Williamson and Shmoys.)
