22C:196 Homework 3

Due: Tuesday, 11/21

Notes: (a) Computer Science Ph.D. students are required to solve Problems 1, 2, 4, 6, and 8. The rest of the students are required to solve the Problems 2, 3, 5, and 7. The problem numbers refer to problems in the textbook, by Mitzenmacher and Upfal. (b) It is possible that solutions to some of these problems are available to you via other textbooks, on-line lecture notes, etc. If you use any such sources, please acknowledge these in your homework *and* present your solutions in your own words. You will benefit most from the homework, if you sincerely attempt each problem on your own first, before seeking other sources. (c) As mentioned in the syllabus, it is *not* okay to discuss these problems with your classmates. But, you are welcome to come and chat with me about the problems. (d) Students who are not Computer Science Ph.D. students will receive extra credit for submitting correct solutions to any non-empty subset of Problems 1, 4, 6, and 8.

1. Earlier in the semester we discussed the following "useful form" of a Chernoff bound: Let $X = \sum_{i=1}^{n} X_i$, where the X_i's are mutually independent 0-1 random variables. Then for all $t > 0$, $\Pr(X > E[X] + t) \leq e^{-2t^2/n}$. Problem 4.13 in the textbook describes a way to prove this claim under the assumption that the X_i's are identically distributed. Solve Problem 4.13 parts (a), (b), and (c).

2. In class we discussed an $\Omega(\sqrt{N})$ lower bound on the running time of the *bit-fixing* algorithm for permutation routing on a hypercube. This motivated the search for a faster randomized alternative to the bit-fixing algorithm. One natural randomized variant of the bit-fixing algorithm is to "fix" the bits (independently for each packet) in a random order. This algorithm is described in Problem 4.22, where you are asked to show that this algorithm is also quite slow. Solve this problem.

3. Problem 6.3.

4. Suppose that $X_1, X_2, \ldots, X_n$ are *negatively associated* random variables. Let $\mathcal{A}$ be a family of disjoint subsets of $\{1, 2, \ldots, n\}$. Show that the random variables $f_A(X_i, i \in A)$ for each $A \in \mathcal{A}$ are also negatively associated, where the f_A's are all non-decreasing or all non-increasing.

5. Problem 6.12.

6. It is known that the function $f(n) = \ln n/n$ is a threshold function for $G_{n,p}$ to have the property of being connected. In fact, the phase transition for connectivity of $G_{n,p}$ is even sharper – if $p = c \ln n/n$ for $c < 1$ then $\Pr(G_{n,p} \text{ is connected}) \to 0$ as $n \to \infty$ and if $p = c \ln n/n$ for $c > 1$ then $\Pr(G_{n,p} \text{ is not connected}) \to 0$ as $n \to \infty$. Problem 6.13 asks you to prove one part of this claim. Solve this problem.

7. Problem 6.18. This is a straightforward application of the Lovász Local Lemma and should provide useful practice.

8. Let $G = (V, E)$ be an arbitrary k-regular directed graph (i.e., every vertex has in- and out-degree k). In this problem, you will show (using the Lovász Local Lemma) that G contains at least $\lfloor k/(3 \ln k) \rfloor$ vertex-disjoint cycles. (The strongest result for this problem seems to be due to Alon (1996) who showed that G contains at least $k/64$ vertex disjoint cycles. It is conjectured that G has $k/2$ vertex disjoint cycles.)

(a) Suppose the vertices of G are partitioned into $c = \lfloor k/(3\ln k) \rfloor$ components chosen independently and uniformly at random. For each vertex v, let A_v be the event that v has no edge to another vertex in its component. Show that $\Pr(A_v) \leq k^{-3}$.

(b) Show that the the collection of events $\{A_v \mid v \in V\}$ has a dependency graph with maximum degree bounded above by $(k+1)^2$.

(c) Deduce from parts (a) and (b) and the LLL that G contains at least c vertex-disjoint cycles.
