22C:137/22M:152 Homework 2

Due: Tuesday, 4/26

Notes: (a) Solve all 8 problems listed below. We will grade some 5-subset of these. (b) It is possible that solutions to some of these problems are available to you via other graph theory books or on-line lecture notes, etc. If you use any such sources, please acknowledge these in your homework. You will benefit most from the homework, if you sincerely attempt each problem on your own first, before seeking other sources. (c) It is okay to discuss these problems with your classmates. Just make sure that you take no written material away from these discussions.

1. Problem 5.32 (iv), Section 5.5, page 35, Schrijver's notes.

2. Problem 5.36, Section 5.5, page 35, Schrijver's notes.

3. Problem 5.37, Section 5.5, page 35, Schrijver's notes.

4. Let $D = (V, A)$ be a directed graph. For any $r \in V$, an *r-arborescence* is a subgraph $D' = (V, A')$ such that (i) $|A'| = |V| - 1$ and (ii) there is a directed path from r to every vertex in V. Show that given a directed graph $D = (V, A)$ and a weight function $w : A \rightarrow \mathbf{R}^+$ there is a polynomial time algorithm to find a mimimum weight arborescence in D.
   Hint: Use the fact that the heaviest common independent set of two weighted matroids can be found in polynomial time.

5. Problem 15, Chapter 5, page 118. This problem leads to a simple algorithm to compute a Δ-coloring for graphs that are not cliques or odd cycles.

6. Problem 27, Chapter 5, page 119.

7. The *greedy algorithm* for graph coloring takes as input a graph G and an ordering σ of the vertices, processes the vertices according to σ, and to each vertex v assigns the smallest available color $i \in \mathbf{N}$.

   (a) Prove that every graph G has a vertex ordering σ such that the greedy algorithm with input G and σ uses $\chi(G)$ colors.

   (b) For all $k \in \mathbf{N}$, inductively construct a tree T_k with maximum degree k and an ordering σ_k of $V(T_k)$ such that greedy algorithm with input T_k and σ_k uses $k + 1$ colors. Note that this shows that the performance ratio of the greedy algorithm may be as bad as $(\Delta(G) + 1)/2$.

8. Let G be the *unit distance graph* in the plane; the vertices are all (infinitely many) points in the plane, with vertices joined by an edge if the Euclidean distance between them is exactly 1. Prove that G is 7-colorable but not 3-colorable.
