22C:131 Homework 3

Due: Wednesday, 4/5

Notes: (a) Solve all 5 problems listed below. The problem numbers refer to problems in the second edition of Sipser. (b) It is possible that solutions to some of these problems are available to you via other theory of computation books or on-line lecture notes, etc. If you use any such sources, please acknowledge these in your homework. You will benefit most from the homework, if you sincerely attempt each problem on your own first, before seeking other sources. (c) It is okay to discuss these problems with your classmates. Just make sure that you take no written material away from these discussions.

1. 8.15 (8.15 in the first edition also).
   Hint. Define a *position* of the game as a triple (c_p, m_p, t) where c_p and m_p are vertices of G denoting the cat's current position and the mouse's current position and t is a boolean variable indicating if it is the cat's turn or not. Then follow these steps:

   (a) Show that there are polynomially many positions.

   (b) The cat has a winning strategy (trivially) in any position in which $c_p = m_p$. Starting with this, identify other positions in which the cat has a winning strategy. To do this you might want to consider a graph whose vertices are the positions and whose (directed) edges go from one position p_1 to another position p_2, if a valid move of the game can transform p_1 to p_2.

   (c) Using this algorithm, you want to determine if the position $(c, m, TRUE)$ has a winning strategy for the cat or not.

2. 8.19 (8.21 in the first edition).

3. 8.23. This is missing from the first edition, so it is stated below.
   Define $UCYCLE = \{\langle G \rangle \mid G \text{ is an undirected graph that contains a simple cycle}\}$. Show that $UCYCLE \in L$. (Note: G may be a graph that is not connected.)
   Hint. If G contains no simple cycle, then it is a forest. The goal is is determine if G is a forest or not. To do this in log space, suppose that G has n vertices and these are arbitrarily labeled 1 through n. For any vertex u, construct a cyclic ordering of the edges incident on u using the labels of the neighbors of u. Here is an example to clarify this. Suppose u has 4 neighbors, labeled 13, 17, 19, and 25. Then the 4 edges incident on u are ordered

   $$\{u, 13\} < \{u, 17\} < \{u, 19\} < \{u, 25\}$$

   with edge $\{u, 13\}$ coming $\{u, 25\}$ since this is a circular order. Having defined this circular order for the edges incident on every vertex u, consider the following traversal algorithm. The algorithm starts with a vertex-edge pair, (u, e), where e is incident on u, and follows e from u to the other endpoint of e. At each step the algorithm enters a vertex on an edge e_1 and leaves the vertex on the edge e_2 that comes after e_1 in the circular order. Show that the algorithm will always return to u. Also show that whether the algorithm returns to u on the edge e or not is important and can be used to deduce something about whether G is a forest or not.

4. *Steve's class* **SC** is defined as the class of languages, each of which can be decided by a deterministic TM that runs in polynomial time *and* uses polylogarithmic space. A TM is said to use polylogarithmic space, if it uses $O(\log^k n)$ space, where k is a constant and n is the size of the input. Use Savitch's Theorem to either show that $\mathbf{NL} \subseteq \mathbf{SC}$ or explain why Savitch's Theorem does not apply here.

   Notes: **SC** was named after Stephen Cook, famous for the Cook-Levin Theorem. For various reasons Steve's class is not as well known as Nick's class, **NC**. **NC** is a complexity class that attempts to categorize problems by virtue of how difficult they are to solve using parallel algorithms.

5. For each of the claims below, write `True`, `False`, or `Unknown`. If you do write `True` or `False`, you need to provide a brief justification. This does not need to be a proof - a proof sketch or a pointer to a theorem - will suffice.

   (a) $\mathbf{L} \subseteq \mathbf{P}$.

   (b) $\mathbf{NL} \subseteq \mathbf{P}$.

   (c) $\mathbf{L}^2 \subseteq \mathbf{NP}$. Here I am using $\mathbf{L}^2$ to denote $SPACE(log^2 n)$.
