**PROBLEM 2.** (1.5 in Arora-Barak)

The TM $\tilde{M}$ that is briefly described in the proof of Claim 1.6 is "almost" oblivious. In fact, the only non-oblivious behavior occurs when the read/write head sweeps back in the right-to-left direction and encounters a "hatted" symbol $\hat{x}$. Suppose that $x$ is from tape $i$, $2 \leq i \leq k$ on the original machine $M$. Depending on whether the read/write head on tape $i$ of the original machine moves left or right in the current step, $\tilde{M}$ might move its head in a different direction and thus $\tilde{M}$ is not oblivious.

To make $\tilde{M}$ oblivious we $\tilde{M}$ could have perform a left-to-right & right-to-left sweep for each of the $k$ tapes of the original machine. This would give $\tilde{M}$ the ability to make changes either to the left or to the right of $\hat{x}$ without having to change directions.

[There are many details left unspecified in this solution, but this level of detail corresponds to the level of detail in the proof of Claim 1.6.]

**PROBLEM 3**

(a) $1, 1, 3, 2, 2, 4, 4$

(b) If $|t_i| = |b_j|$ for some $1 \leq i \leq k$, halt with output 1.
If $|t_i| > |b_j|$ & $|t_j| < |b_j|$ for some $1 \leq i \neq j \leq k$, halt with output 1.
Otherwise (i.e., $|t_i| > |b_j| + i: 1 \leq i \leq k$ or $|t_i| < |b_i| + i: 1 \leq i \leq k$) halt with output 0.
PROBLEM 4.
To show that $R$ is uncomputable, we reduce ACCEPT to $R$.

**ALGO. for ACCEPT**
**INPUT**: $<x,x> \in \{0,1\}^* \times \{0,1\}^*$

1. Create a TM $M'$ by modifying $M_x$ that works as follows:
   On input $w$:
   - if $w \neq "01"$ and $w \neq "10"$ then reject (i.e., output 0)
   - if $w = "01"$ accept (i.e., output 1)
   - if $w = "10"$ simulate $M_x$ on $x$. If $M_x$ accepts $x$, then accept; if $M_x$ halts and rejects $x$, then reject.

2. Compute $R(L(M'))$

3. If $R(L(M')) = 1$ then output 1, else output 0.

To see the correctness of the reduction note that if $<x,x> \in$ ACCEPT then $L(M') = \{ "01", "10" \}$ and if $<x,x> \notin$ ACCEPT, $L(M') = \{ "01" \}$.
In the former case $R(L(M'))$ will evaluate to 1 and in the latter case $R(L(M'))$ will evaluate to 0.

PROBLEM 5.
To show that $B$ is uncomputable, we reduce HALT to $B$.

**ALGO for HALT**
**INPUT**: $<x,x> \in \{0,1\}^* \times \{0,1\}^*$

1. Create a TM $M'$ by modifying $M_x$ that works as follows:
   On input $w$:
   - Simulate $M_x$ on $x$
   - if $M_x$ halts on $x$ then write a "0" on an extra tape that is designated the output tape for $M'$

2. Compute $B(L(M'))$.

3. If $B(L(M')) = 1$ then output 1, else output 0.

Correctness follows from the fact that $B(L(M')) = 1$ iff $M_x$ halts on $x$. 


**Problem 7.** (1.12 in Arora-Barak)

(a) For any \(x \in \{0,1\}^*\), let \(M^x = \{ TM \mid M \text{ halts on } x \}\). Let \(S^x\) denote the set of all partial functions computed by TMs in \(M^x\). Clearly, \(S^x\) is non-trivial since it is non-empty and there are partial functions computable by TMs that lie outside \(S^x\). Thus by Rice's Theorem \(f_{S^x}\) is not computable.

For \(x \in \{0,1\}^*\), define a function \(HALT^x : \{0,1\}^* \rightarrow \{0,1\}\) as:

\[
HALT^x(x) = \begin{cases} 
1 & \text{if } M^x \text{ halts on } x \\
0 & \text{otherwise}
\end{cases}
\]

Note that \(HALT^x(x) = f_{S^x}(x)\) for all \(x \in \{0,1\}^*\). Hence, \(HALT^x\) is also not computable.

Now we reduce \(HALT^x\) to \(HALT^y\) by noting that \(HALT^x(x) = HALT(y, x)\) for all \(x \in \{0,1\}^*\). Thus if \(HALT\) were computable, \(HALT^x\) would be as well—a contradiction.

(b) Rice's Theorem.

**Proof:** Consider a nontrivial set \(S\) of partial functions. Without loss of generality suppose that \(\emptyset\) (the function that is not defined anywhere) is not in \(S\). Also, since \(S\) is not empty it contains a function \(f^*\) that is defined on some \(x \in \{0,1\}^*\) and can be computed by a TM—say \(M^f\).

Now we can reduce \(HALT^x\) to \(S\) as follows.

**Algorithm for \(HALT^x\)**

- **Input:** \(x \in \{0,1\}^*\)

  1. Create a TM \(M'\) by modifying \(M^x\) that works as follows:
     - On input \(w \in \{0,1\}^*\), \(M'\) runs \(M^x\) on \(x\); if \(M^x\) halts, \(M'\) simulates \(M^f\) on \(w\).

  2. Compute \(f_S(LM')\).

  3. If \(f_S(LM') = 1\) then output 1, else output 0.

To see the correctness of the reduction note that if \(f_S(LM') = 1\) then it must be that \(M^x\) halts on \(x\) and if \(f_S(LM') = 0\), it doesn't.