22m:033 Notes:
2.2 Inverse of a Matrix

Dennis Roseman
University of Iowa
Iowa City, IA

http://www.math.uiowa.edu/∼roseman

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1 Definition of Inverse

Definition 1.1 *An* $n \times n$ *matrix* A ***is invertible*** *if there exists an* $n \times n$ *matrix* C *such that*

$$AC = I_n \text{ and } CA = I_n.$$

If C *exists it is called the* ***inverse of*** A *and denoted by* A^{-1}*.*

Remark 1.2 If an inverse exists, it is unique.

With our notation we can express the equations in Definition 1.1 as:

$$AA^{-1} = I_n \text{ and } A^{-1}A = I_n.$$

A matrix that is not invertible is called a **singular matrix**.

A matrix that is invertible is called a **nonsingular matrix**.

Example 1.3 We can check that the inverse of $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ is $\begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$

by multiplying the two matrices.

For a 2×2 matrix we can easily explain invertibility (Theorem 4 of text):

Proposition 1.4 *If* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *then* A *is invertible if and only if* $ad - bc \neq 0$*. In fact if* $ad - bc \neq 0$

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

The number $ad - bc$ is called the **determinant of** A.

Note that we could have calculated the inverse in Example 1.3 using these formulas. Note that the determinant of $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ is -2.

Remark 1.5 WARNING: The simple formula for the inverse of a 2×2 **matrix only works for a** 2×2 **matrix.** We will find an extension of this formula later in the course but it is not simple. In fact find better ways of calculating inverses in this section.

2 Using inverse to solve equations

This is Theorem 5 in text:

Proposition 2.1 *If A is an $n \times n$ invertible matrix then for any $\overrightarrow{b} \in R^n$ the equation $A\overrightarrow{x} = \overrightarrow{b}$ has the unique solution $\overrightarrow{x} = A^{-1}\overrightarrow{b}$.* ∎

3 Properties of inverse

These are listed in Theorem 6 in text

Proposition 3.1

- *If A is invertible then so is A^{-1} and $(A^{-1})^{-1} = A$*
- *If A and B are invertible $n \times n$ matrices, then AB is also invertible and*

$$(AB)^{-1} = B^{-1}A^{-1}$$

- *If A is invertible then so is A^T and*

$$(A^T)^{-1} = (A^{-1})^T \quad ∎$$

4 Elementary Matrices

An elementary matrix allows us to express the process of a row operation by matrix multiplication.

Definition 4.1 *An **elementary matrix** is one obtained from an identity matrix by performing a single row operation.*

Proposition 4.2 *If we obtain B from A by a single row operation, and if E is the elementary matrix corresponding to this row operation. Then $B = EA$*

Example 4.3 Suppose we take $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ and add twice the first row to the second row. We then get $B = \begin{pmatrix} 1 & 2 & 3 \\ 6 & 9 & 12 \end{pmatrix}$.

The corresponding elementary operation is $E = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ and we can check that

$$B = EA. \quad \blacksquare$$

This is Theorem 7 of the text

Proposition 4.4 *An $n \times n$ matrix A is invertible if and only if it is row equivalent to I_n. Also any sequence of row operations that reduces A to I_n also transforms I_n to A^{-1}.*

Remark 4.5 For the simple case that *one* row operation reduces A to I_n then it also transforms I_n to A^{-1}. Let E the the corresponding elementary matrix. Then

$$EA = I$$

Take the inverse of both sides and use the fact that $I^{-1} = I$ we get

$$A^{-1}E^{-1} = (EA)^{-1} = I^{-1} = I$$

So

$$(A^{-1}E^{-1})E = IE$$

or

$$A^{-1} = A^{-1}I = (A^{-1}(E^{-1})E) = IE = E \quad \blacksquare$$

5 Finding inverses, if they exist

Remark 5.1 Take a square matrix A and make an augmented matrix $M = (AI)$. Row reduce M.

IF M has the form $M = (IB)$, then A does have an inverse and $A^{-1} = B$. IF M ends up with a row of zeros then A does not have an inverse.

Example 5.2 We can calculate that the inverse of $\begin{pmatrix} 1 & -1 & 1 \\ 1 & 2 & 3 \\ 3 & -2 & 1 \end{pmatrix}$ is $\begin{pmatrix} -1 & \frac{1}{8} & \frac{5}{8} \\ -1 & \frac{1}{4} & \frac{1}{4} \\ 1 & \frac{1}{8} & -\frac{3}{8} \end{pmatrix}$ since when we row reduce

$$\begin{pmatrix} 1 & -1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 3 & -2 & 1 & 0 & 0 & 1 \end{pmatrix}$$

we get

$$\begin{pmatrix} 1 & 0 & 0 & -1 & \frac{1}{8} & \frac{5}{8} \\ 0 & 1 & 0 & -1 & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 1 & 1 & \frac{1}{8} & -\frac{3}{8} \end{pmatrix}. \blacksquare$$

Example 5.3 On the other hand we deduce that $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ has no inverse since when we row reduce

$$\begin{pmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 4 & 5 & 6 & 0 & 1 & 0 \\ 7 & 8 & 9 & 0 & 0 & 1 \end{pmatrix}$$

we get

$$\begin{pmatrix} 1 & 0 & -1 & 0 & -\frac{8}{3} & \frac{5}{3} \\ 0 & 1 & 2 & 0 & \frac{7}{3} & -\frac{4}{3} \\ 0 & 0 & 0 & 1 & -2 & 1 \end{pmatrix}$$

6 Problems

Question 6.1 Let $M_\theta = \begin{pmatrix} \cos\theta & \sin(-\theta) \\ \sin\theta & \cos\theta \end{pmatrix}$. Show, using Definition 1.1 that $M_\theta^{-1} = M_{-\theta}$.

Question 6.2 Following the argument of Remark 4.5 show that For the simple case that *two* row operations reduces A to I_n then they also transform I_n to A^{-1}.

Question 6.3 Use the method of Remark 5.1 to derive the formula for the inverse given in Proposition 1.4.