22m:033 Notes:
1.3 Vector Equations

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1 Algebra and Geometry

We think of geometric things as subsets of the plane or of three dimensional space. Points are either ordered pairs of numbers or ordered triples. Linear algebra gives us a simple and powerful way of thinking about these points as algebraic so that we can "add points" and do multiplications.

2 n-space

Definition 2.1 *We define real n-dimensional space to be the set of ordered n-tuples of real numbers and denote this by R^n.*

Remark 2.2 Often we will denote a point of R^n by $(x_1, x_2, \ldots, x_n)$.

1. So R^1, also denoted R is the set of **real numbers**.
2. And R^2 is all ordered pairs of real numbers (x_1, x_2), sometimes simply called **the plane** or **2-dimensional space**.

3. And R^3 is all ordered triples of real numbers (x_1, x_2, x_3), sometimes simply called **the space**, or 3**-dimensional space**.
4. And R^4 is all ordered 4-tuples of real numbers (x_1, x_2, x_3, x_4), is called 4**-dimensional space**.
5. Also R^5 is all ordered 5-tuples of real numbers $(x_1, x_2, x_3, x_4, x_5)$, is called 5**-dimensional space**.
6. ... and on and on ....

Remark 2.3 When we talk about "the" plane, we generally will be referring to R^2 rather that "a" plane in R^3.

3 Vectors in R^n

Definition 3.1 *A* ***vector*** *in R^n is an n-tuple, called an n-**dimensional vector**, of numbers which we write as a matrix of one column.*

So we represent the n-tuple $(x_1, x_2, \ldots, x_n)$ by $\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$.

The number x_i is called the **i-th component of the vector**. The number n is called the **dimension of the vector**.

In these notes, on the blackboard, and in tests we use the notation $\overrightarrow{v}$ to indicate v is a vector (the text uses **bold font**).

Two vectors are equal if they have the same dimension and components of one equal components of the other:

Definition 3.2 *Two* ***vectors***

$$\overrightarrow{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix} \ \textit{and}\ \overrightarrow{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

are equal *if* $m = n$ *and* $u_i = v_i$ *for* $1 \leq i \leq n$.

Remark 3.3 A vector is a $1 \times n$ matrix. But since there is only one column, we will not use the double subscript notation. Instead of writing

$$\overrightarrow{v} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \text{ instead of } \begin{pmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{pmatrix}$$

Another way to write this is to say $\overrightarrow{v} = (x_i)$ where $1 \leq i \leq n$.

3.1 Addition of vectors

Two vectors of the same dimension n can be added:

Definition 3.4 *If* $\overrightarrow{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$ *and* $\overrightarrow{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$ *Then the* ***vector sum*** $\overrightarrow{u}+\overrightarrow{v}$ *is also an* n*-dimensional vector defined:* $\overrightarrow{u} + \overrightarrow{v} = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{pmatrix}$.

Example 3.5 For example

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \begin{pmatrix} a \\ b \\ 3 \end{pmatrix} = \begin{pmatrix} a+1 \\ b+2 \\ 6 \end{pmatrix}.$$

Definition 3.6 *A vector is called a* ***zero vector*** *if all of its coordinates are zero. Where the dimension is understood we will use the notation* $\overrightarrow{0}$ *for the zero vector.*

3.2 Scalar multiplication

Definition 3.7 *If* $\overrightarrow{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$ *and* c *is any number then the* ***scalar product*** $c\overrightarrow{u}$ *is the vector defined by:*

If $c\overrightarrow{u} = \begin{pmatrix} cu_1 \\ cu_2 \\ \vdots \\ cu_n \end{pmatrix}$.

Remark 3.8 NOTE CAREFULLY: For addition we add two vectors (of same dimension) and get another vector (of that dimension). With a scalar product we combine *two very different things*: a vector and a number—what we get is a vector.

Example 3.9

$$2\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}, -1\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ -3 \end{pmatrix}, \text{ and } 0\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Definition 3.10 *If* $\overrightarrow{u}$ *is any vector* $-\overrightarrow{u} = (-1)\overrightarrow{u}$.

3.3 Graphic view of vector in the plane and space

Graphically, we depict a vector in the plane or space as an arrow (except the zero vector). There are two ways to do this. In the plane the **origin** is $(0, 0)$, in space $(0, 0, 0)$

- (Based arrow method) Depict the vector as an arrow with point of arrow at point whose coordinates are the vector coordinates and with the "tail" point of the arrow at the origin.
- (Free vector method) Depict the vector as any arrow which is parallel to that obtained from the based arrow method.

Remark 3.11 The text uses only the based arrow method. In multi-variable calculus, physics and many kinds of engineering, you need the other as well.

3.4 Geometric view of vectors in R^2

Given two non-zero vectors $\overrightarrow{u} = \begin{pmatrix} a \\ b \end{pmatrix}$ and $\overrightarrow{v} = \begin{pmatrix} c \\ d \end{pmatrix}$ we can describe the sum $\overrightarrow{u} + \overrightarrow{u} = \overrightarrow{v} = \begin{pmatrix} a+c \\ b+d \end{pmatrix}$ as follows:

- (Based arrow method) Assume the points $(0,0)$, (a,b) and (c,d) do not lie in a line. Then these three points determine a unique parallelogram P with one angle α at $(0,0)$. Then the point $(a+c, b+d)$ will be the vertex point of P diagonally opposite the vertex of α. In other words $\overrightarrow{u} + \overrightarrow{u}$ is represented by this diagonal.

- (Free vector method) Assume the free vectors do not lie in parallel lines. Draw $\overrightarrow{u}$ as an arrow from a point A to point B and draw $\overrightarrow{v}$ as a free vector which starts at B and goes to C. The vector $\overrightarrow{u} + \overrightarrow{u}$ can be depicted as an arrow that goes from A to C.

Remark 3.12 It is often better to use the free method when dealing with a sum of three vectors or more.

3.5 Geometric view of vectors R^3

Three points in space, if they do not lie on a line, determine a plane H. The two vectors correspond to arrows that "lie" in that H. Once we have this we use either of the methods described in Section 3.4.

3.6 Vectors in R^4

It is difficult for many to visualize R^4, however addition of 4-dimensional vectors only really needs visualization in two dimensions.

In later sections we will be able to articulate and show that three points in R^4, if they do not lie on a line, determine a two dimensional subset H which is "just like a plane" . The two vectors correspond to arrows that "lie" in that H. Once we have this we use either of the methods described in Section 3.4.

4 Algebraic Properties of Vector Addition and Scalar Multiplication

Proposition 4.1 *Suppose* $\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w}$ *are vectors in* R^n *and* c, d *are numbers.*

1. $\overrightarrow{u} + \overrightarrow{v} = \overrightarrow{v} + \overrightarrow{u}$
2. $(\overrightarrow{u} + \overrightarrow{v}) + \overrightarrow{w} = \overrightarrow{u} + (\overrightarrow{v} + \overrightarrow{w})$
3. $\overrightarrow{u} + \overrightarrow{0} = \overrightarrow{0} + \overrightarrow{u} = \overrightarrow{u}$
4. $\overrightarrow{u} + (-\overrightarrow{u}) = -\overrightarrow{u} + \overrightarrow{u} = \overrightarrow{0}$
5. $c(\overrightarrow{u} + \overrightarrow{v}) = c\overrightarrow{u} + c\overrightarrow{v}$
6. $(c + d)\overrightarrow{u} = c\overrightarrow{u} + d\overrightarrow{u}$
7. $c(d\overrightarrow{u}) = (cd)\overrightarrow{u}$
8. $1\overrightarrow{u} = \overrightarrow{u}$

Remark 4.2 We will write $\overrightarrow{u} + (-\overrightarrow{u})$ as $\overrightarrow{u} - \overrightarrow{u}$.

5 Linear Combinations

Definition 5.1 *Given a set of vectors* $\overrightarrow{v_1}, \overrightarrow{v_1}, \ldots, \overrightarrow{v_p}$ *and numbers* $c_1, c_2, \ldots c_p$ *the vector*

$$c_1\overrightarrow{v_1} + c_2\overrightarrow{v_1} + \cdots + c_p\overrightarrow{v_p}$$

*is called a **linear combination** of the vectors* $\overrightarrow{v_1}, \overrightarrow{v_1}, \ldots, \overrightarrow{v_p}$ *with **weights*** $c_1, c_2, \ldots c_p$

Example 5.2 Let us see if $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ is a linear combination of $\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$

If it is a linear combination, we can find numbers c_1 and c_2 such that

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = c_1 \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

this means that

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = c_1 \begin{pmatrix} -c_1 + 2c_2 \\ c_1 + c_2 \\ 2c_1 + c_2 \end{pmatrix}$$

The given vector is a linear combination if we can solve the following set of equations:

$$\begin{aligned} -c_1 + 2c_2 &= 1 \\ c_1 + c_2 &= 2 \\ 2c_1 + c_2 &= 3 \end{aligned}$$

We use our methods to solve this system. The augmented matrix is:

$$\begin{pmatrix} -1 & 2 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \end{pmatrix}$$

The row reduced echelon form of this is

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

and we conclude that the given vector is a linear combination if we use weights $c_1 = 1, c_2 = 1$.

6 The span of a set of vectors

Definition 6.1 *Given vectors $\overrightarrow{v_1}, \overrightarrow{v_1}, \ldots, \overrightarrow{v_p}$ in R^n, the **span of** $\overrightarrow{v_1}, \overrightarrow{v_1}, \ldots, \overrightarrow{v_p}$ (or the **subset of** R^n **spanned by** $\overrightarrow{v_1}, \overrightarrow{v_1}, \ldots, \overrightarrow{v_p}$) is the set of all linear combinations of the vectors $\overrightarrow{v_1}, \overrightarrow{v_1}, \ldots, \overrightarrow{v_p}$.*

We denote this by $Span\{\overrightarrow{v_1}, \overrightarrow{v_1}, \ldots, \overrightarrow{v_p}\}$.

Example 6.2 The calculation of Example 5.2 shows that $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ is in the span of $\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$.

On the other hand, $\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$ is in *not* in the span of $\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$.

This can be seen since when we row reduce

$$\begin{pmatrix} -1 & 2 & 3 \\ 1 & 1 & 2 \\ 2 & 1 & 1 \end{pmatrix}$$

we get

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

.

This means there are no solutions to the vector equation

$$\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = c_1 \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$