Fast Stochastic AUC Maximization with $O(1/n)$ Convergence Rate

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1. Problem: We consider statistical learning with AUC (Area under the ROC curve) maximization where one random data is received at one iteration for updating the model.

2. Definition: AUC(h) = Pr(h(x) ≥ h(y)|y = 1, y' = −1), where (x, y, (x', y')) ∈ $\mathbb{R}^2 \times (-1, 1)$, h: $\mathbb{R}^d \rightarrow \mathbb{R}$ is score function.

3. Challenges: Online AUC Optimization is challenging since AUC loss depends on pairs of examples. The pairwise nature in the definition of AUC makes it difficult to design algorithms suitable for the classical stochastic or online setting.

4. Main Contribution: Building on a saddle point formulation of AUC, we developed a novel stochastic (online) algorithm with $O(1/n)$ convergence rate for AUC maximization, where n is the number of received examples (this is the first result of fast rate for AUC).

Related Work

1. Zhao et al. [1] designed an online algorithm which keeps a large buffer (with size $O(\sqrt{n})$) and then utilizes the data in the buffer to update the classifier.

2. Gao et al. [2] designed an online algorithm which needs to keep the first and second order statistics.

3. Ying et al. [3] reformulates the original problem into a convex-concave saddle point problem, and then utilize the standard stochastic gradient method to solve it.

Saddle Point Reformulation of AUC Maximization [3]:

$$\min_{w \in \mathbb{R}^d} \max_{\alpha \in \mathbb{R}^d} \{F(w, a, b, \alpha) = \mathbb{E}_a[F(w, a, b, \alpha, x)]\},$$

where $x = (x, y), F(w, a, b, \alpha, x) = (1 - p)(w'x - \alpha)^2I_{y=1} + p(w'x - b)^2I_{y=1} + p(1 - p)^2(1 + 2 \alpha)(p(w'x - b)^2 - (1 - p)^2(w'x - b)^2I_{y=1})$.

Key Observation

Define $v = (w, a, b)$, assume sup$_{a \in \mathbb{R}^d}$ $\mathbb{E}_a[I_{y=1}] \leq \kappa, \Omega_0 = \{v \in \mathbb{R}^d: \|v\|_1 \leq R, \|a\| \leq R, |b| \leq R, \}, \Omega_2 = \{v \in \mathbb{R}^d: \|v\|_1 \leq 2 \kappa R\}$. $f_1(v) = \max f(v, \alpha)$ restricted on the set $\Omega_2$ satisfies a quadratic growth condition, i.e., for any $v \in \Omega_1$, there exists $c > 0$ such that $\|v - v^*\| \leq c(f_1(v) - \min_{w \in \Omega_2} f_1(v))^{1/2}$.

Algorithm and Theoretical Results

1. Standard Primal-Dual Stochastic Gradient Algorithm (PDSG)

   **Algorithm 1 PDSG** $(v_t, \alpha_t, r, D, T, y)$

   1. Initialize variables $A_t \in \mathbb{R}^{d 	imes d}, A_t \in \mathbb{R}^{d 	imes d}, T_t, T, r \in \mathbb{R}$ as zeros
   2. for $t = 1, \ldots, T$
   3. Receive a sample $x_t = (x_t, y_t)$
   4. Update $A_t, T_t, r$ using the data $x_t$
   5. $v_{t+1} = \Pi_{\Omega_0} (\alpha_t, \beta_t) (v_t - \eta_t \beta_t F(v_t, \alpha_t, z_t))$
   6. $\alpha_{t+1} = \Pi_{\Omega_0} (\alpha_t, \beta_t) (\alpha_t + \eta_t \beta_t F(v_t, \alpha_t, z_t))$
   7. end for
   8. Compute $\bar{v}_T = \frac{1}{T} \sum_{t=1}^T v_t$ and $\bar{\sigma} = (\frac{1}{T} - \frac{1}{T-1})^\sigma \bar{v}_T$
   9. Let $r = r/\sqrt{T}$, update $\beta, D$
   10. return $(\bar{v}_T, \bar{\sigma}, \beta, r, D)$

   **Remark:** $A_t$ store the summation of feature vectors for positive (negative) examples, $T_t$ stand for the number of received positive (negative) examples until the current iteration.

   **Theoretical Guarantee of PDSG**

   Suppose $\|v_t - v^*\| \leq \eta$, where $v^* \in \Omega_2$ is the optimal solution closest to $v_t$, run the PDSG for $T$ iterations. Then with probability at least $1 - \delta$, $\max f(v_T, \alpha) - \min \max f(v, \alpha) \leq O\left(\frac{\ln(T/\delta)}{\sqrt{T}}\right)$.

2. Motivated by Juditsky and Nesterov [4], we design Fast Stochastic AUC Maximization Algorithm (FSAUC)

   **Algorithm 2 FSAUC**

   1. Set $m = \lceil \log_2 \frac{m_0}{\ln(n/m_0)} \rceil - 1, m_0 = \lceil n/m \rceil, R_0 = 2\sqrt{1 + 2\kappa^2 R}, \beta_0 = 1 + 8\kappa^2$
   2. Initialize $\bar{v}_0 = 0 \in \mathbb{R}^{d 	imes d}, \bar{\alpha}_0 = 0, \bar{v}_1 = 0$
   3. for $k = 1, \ldots, m$
   4. Set $\eta_k = \frac{1}{\sqrt{k}} R_{k-1}$
   5. $(\bar{v}_k, \bar{\alpha}_k, \beta_k, R_k, D_k) = PDSG(\bar{v}_{k-1}, \bar{\alpha}_{k-1}, R_{k-1}, D_{k-1}, m_0, \eta_k)$
   6. end for
   7. return $\bar{v}_m$

   **Main Theorem**

   **Theoretical Guarantee of FSAUC**

   When $n > \max \{100, m_0 \log(m_0)\}$, then with probability at least $1 - \delta$, $\max f(\bar{v}_m, \alpha) - \min \max f(v, \alpha) \leq O\left(\frac{1}{\delta} \log(1/\delta)/n\right)$.

References: