Market Equilibrium via the Excess Demand Function

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ABSTRACT

We consider the problem of computing market equilibria and show three results. (i) For exchange economies satisfying weak gross substitutability we analyze a simple discrete version of tâtonnement, and prove that it converges to an approximate equilibrium in polynomial time. This is the first polynomial-time approximation scheme based on a simple tâtonnement process. It was only recently shown, using vastly more sophisticated techniques, that an approximate equilibrium for this class of economies is computable in polynomial time. (ii) For Fisher's model, we extend the frontier of tractability by developing a polynomial-time algorithm that applies well beyond the homothetic case and the gross substitutes case. (iii) For production economies, we obtain the first polynomial-time algorithms for computing an approximate equilibrium when the consumers' side of the economy satisfies weak gross substitutability and the producers' side is restricted to positive production.

Categories and Subject Descriptors

F.2.2 [**Analysis of Algorithms and Problem Complexity**]: Nonnumerical Algorithms and Problems—*Geometrical problems and computations*

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1. INTRODUCTION

The market equilibrium problem consists of finding a set of prices and allocations of goods to economic agents such that each agent maximizes her utility, subject to her budget constraints, and the market clears. The equilibrium equations, which are satisfied under mild assumptions [1], express a static condition characterized by the fact that the market demand for each good equals its market supply. This notion does not predict any kind of dynamics leading to an equilibrium, although it conveys the intuition that, in any process leading to a stable state where demand equals supply, a disequilibrium price of a good will have to increase if the demand for such a good exceeds its supply, and viceversa. The proofs of existence of equilibrium [22] use general fixed point theorems and therefore do not tell us how an equilibrium can be efficiently computed. An important question that theoretical computer scientists have begun to address is whether there are efficient algorithms for computing equilibria.

Tâtonnement. In 1874 Léon Walras introduced a price-adjustment mechanism, which he called *tâtonnement* [32]. He took inspiration from the workings of the stock-exchange in Paris, and suggested a trial-and-error process run by a fictitious *auctioneer*. The economic agents receive a price signal, and report their demands at these prices to the auctioneer. The auctioneer then adjusts the prices in proportion to the magnitude of the aggregate excess demands, and announces the new prices. In each round, agents recalculate their demands upon receiving the newly adjusted price signal and report these new demands to the auctioneer. The process continues until prices converge to an equilibrium.

In its continuous version, as formalized by Samuelson [31], the tâtonnement process is governed by a system of differential equations over variables that represent prices:

$$\frac{d\pi_k}{dt} = G_k(Z_k(\pi)), \;\; k = 1, 2, \ldots, n, \tag{1}$$

where $G_k()$ is some continuous and sign-preserving function, and $Z_k()$ is the market excess demand function (see Section 2 for the definition of excess demand and related concepts).

Contrary to Walras' intuition and hopes (see [32], p.172), tâtonnement does not converge in general, but only for markets satisfying certain restrictions (see the work by Hahn [14], Negishi [24, 25], Arrow, Block, and Hurwicz [2], and Arrow and Hurwicz [3, 4], and the review by Hahn [15]).

A central assumption is that of gross substitutability (GS). A market is said to satisfy GS (resp., weak gross substitutability – WGS) if increasing the prices for some of the goods while keeping some others fixed can only cause an increase (resp., cannot cause a decrease) in the aggregate demand for the goods whose price is fixed.

Arrow, Block, and Hurwicz [2] showed that the continuous process (1) is convergent for markets satisfying GS. Their result was extended to prove convergence when only WGS holds. This convergence result raises the question of whether there exist simple polynomial-time algorithms corresponding to a discrete version of tâtonnement. Our first result answers this question in the affirmative.

We present a simple algorithm, which is a discrete version of tâtonnement, and prove that it converges to an *approximate equilibrium* in polynomial time for exchange markets satisfying WGS. We consider the following discrete version of the continuous process (1):

$$\pi_j^i = \pi_j^{i-1} + \alpha Y_j^{i-1}, \tag{2}$$

where π_j^k denotes the price of good j at the k-th iteration, Y_j^k (an approximation to) the market excess demand for good j when its price is π^k, and α is a suitably chosen parameter that does not depend on i, but depends on the number of goods and on the approximation parameters (see below).

To be more precise, we first "transform" our original market into another one for which an equilibrium price is guaranteed to belong to a region where the price ratios are nicely upper bounded. We then prove that it is sufficient to compute an approximate equilibrium in the transformed market. We compute a sequence of prices in the transformed market by applying the adjustment rule (2). If, at any given step, the adjustment rule (2) produces a price vector which falls outside the region, then, instead of applying rule (2), we update the price by returning the closest price vector within the region.

We analyze the convergence of this algorithm, and prove that it terminates after a number of steps which is polynomial in the size of the market, i.e., in the number of traders plus the number of goods plus the number of bits needed for encoding the rational numbers that describe the utility functions and initial endowments. The dependence of the running time on the approximation parameter ε is polynomial in $1/\varepsilon$.

There has been previous work that shows that a discrete version of tâtonnement converges to an equilibrium (see for example [28]), but to the best of our knowledge, convergence in polynomial time has not been established. Several recent algorithms for the computation of market equilibria are iterative methods that can be seen as versions of tâtonnement. Garg and Kapoor introduced a simple algorithm with an auction interpretation for the exchange model with linear utilities, and which can be viewed as a tâtonnement scheme [12]. This approach was generalized in [13] to handle additively separable utility functions satisfying WGS.

Jain et al. [18] introduced two algorithms to approximate the equilibrium in an exchange economy with linear utilities. Their second algorithm, which uses a black box to compute the equilibrium in the Fisher's model[1], runs in polynomial time. The algorithm also generalizes to utility functions that satisfy WGS, provided a polynomial-time algorithm for computing an equilibrium in Fisher's model with such utility functions is available. They conjectured that their first algorithm, where the price adjustment rule follows the principle of increasing the price of the goods for which the demand exceeds the supply, runs in polynomial time.

Note that our result applies to more general scenarios than the previous work mentioned above, and that the algorithm is extremely simple and natural. We only assume WGS of the excess demand, and do not assume separability of the utility functions or access to any solver for Fisher's model. Indeed, our simple algorithm computes an approximate equilibrium under precisely the same assumptions as in [8], where a polynomial-time algorithm based on the Ellipsoid Algorithm was developed. Our proof of convergence is based on a refinement of the machinery developed in [8], and in part inspired by some experimental results obtained in [6] on different versions of tâtonnement. Due to its simplicity, versions of tâtonnement have been implemented and experimentally analyzed. We refer the reader to the experimental results in [6] and the references therein.

Fisher's model. The second result of this paper applies to Fisher's model. In this setting, it is known how to compute equilibrium prices and allocations in polynomial time when the utility functions are homothetic [11, 9] or when they satisfy WGS [8]. We introduce a polynomial-time algorithm that applies to markets where the traders have *monotone demand functions*, a class which contains many utility functions not covered by previous work (see below). In the spirit of [8], we provide computationally tractable proofs of separations that guarantee that the Ellipsoid Algorithm returns an approximate equilibrium in polynomial time.

The demand $x()$ of a trader is said to be *monotone* if for any pair of distinct positive prices $\pi, \pi' \in \mathbf{R}_+$, we have $(\pi - \pi') \cdot (x(\pi) - x(\pi')) \leq 0$. This property is an expression of the *law of demand* (see [21]).

Mitiushin and Polterovich [23] introduced the following sufficient condition on a twice-differentiable, concave, and monotonically increasing utility function u that implies the monotonicity of demand:

$$\sigma(x) = -\frac{x \cdot \partial^2 u(x) x}{x \cdot \partial u(x)} < 4 \text{ for all } x.$$

To verify that monotonicity is a significant extension of homotheticity and that it also contains several families of utilities not satisfying WGS, note that

- If $u()$ is homogeneous of degree one[2], then $\sigma(x) = 0$ for all x;
- if $u(x)$ is an additively separable utility function of the form $\sum_j u_j(x_j)$, the sufficient condition for gross substitutability (which is used in [13]) is that $x_j u_j'(x_j)$ be non-decreasing. This translates into a non-negativity condition on the derivative of $x_j u_j'(x_j)$, which yields $-[x_j u_j''(x_j)/u_j'(x_j)] \leq 1$. In contrast, the sufficient

[1]Fisher's model is a market of n goods desired by m utility maximizing buyers with fixed incomes.

[2]A utility function $u(\cdot)$ is *homogeneous* (of degree one) if it satisfies $u(\alpha x) = \alpha u(x)$, for all $\alpha > 0$. A *homothetic* utility function is an increasing monotonic transformation of a homogeneous utility function. Linear utility functions, and the more general CES functions (see [8] for definitions) are examples of homogeneous utility functions.

condition of Mitiushin and Polterovich for monotonicity becomes

$$-[x_j u_j''(x_j)/u_j'(x_j)] < 4.$$

Our result for the Fisher setting therefore encompasses a lot of interesting cases not handled by previous methods [11, 18, 12, 9, 13, 5, 8].

Production. Our third contribution is a polynomial time algorithm to compute an approximate market equilibrium for economies with production, provided the consumers' side of the economy satisfies a gross substitutability condition. We obtain this result by weaving together a "reduction" by Primak [28] of production to exchange and a lemma in [8] that applies to the exchange model. In our result, we adopt the production model used by Nenakov and Primak [26], Primak [28], and Newman and Primak [27]. Their model of an economy is a version of the model used by Arrow and Debreu [1], with the restriction that production sets be constrained to the positive orthant.

In the work of Arrow and Debreu [1], each production plan is a vector that has positive elements, corresponding to outputs of production, and negative elements, corresponding to inputs. The model that we consider only allows vectors with nonnegative elements. Nevertheless, there are scenarios that our model captures. For instance, each firm is initially endowed with a bundle of inputs to its production technology. This bundle then determines the vectors of consumable outputs that can be produced using the technology. The vectors of consumable goods form the production set of the firm. At a given price vector for the consumable goods, each firm supplies the vector of consumable goods that maximizes its profit, and each consumer demands the vector of consumable goods that maximizes her utility subject to her budget constraint. At equilibrium, demand equals supply for the consumable goods.

If we allow production technologies to be arbitrary convex sets, then multiple disconnected equilibria can appear even if the utility functions satisfy gross substitutability – see the construction in [21] with two consumers having linear utility functions. In the scenarios to which the algorithms in this paper, and indeed all previous polynomial-time algorithms, apply, there cannot be multiple disconnected equilibria.

Jain et al. [19] developed polynomial-time algorithms for a model that includes production, but assumes that the incomes of the consumers are fixed and do not depend on the prices. The work of Nenakov and Primak [26] and Jain [16, 17] handles a certain class of utility functions, but the relationship of this class to gross substitutability is far from clear. Newman and Primak [27] described an Ellipsoid Algorithm for a model where the consumers have linear utility functions. As discussed in [8], their algorithm does not guarantee an approximate equilibrium in polynomial time.

All three results of this paper rely on an oracle for computing the excess demand function at a given price in polynomial time. Since the computation of the excess demand at a given price corresponds to solving an explicit convex program, such an oracle is usually easy to construct. Furthermore, note that in many applications the equilibrium problem is directly presented in terms of the aggregate excess demand.

Organization of this abstract. In Section 2 we introduce and analyze a discrete version of the tâtonnement process. We prove that it runs in polynomial time. In Section 3 we present our algorithm for Fisher's model, and in Section 4 we describe a model of production and prove related computational results.

2. A TÂTONNEMENT ALGORITHM

We first describe the exchange market model and provide some basic definitions. Let us consider a market M with m economic agents who represent traders of n goods. Let $\mathbf{R}^n_+$ denote the subset of $\mathbf{R}^n$ with all nonnegative coordinates. The j-th coordinate in $\mathbf{R}^n$ will stand for good j. Each trader i has a concave, nonsatiable[3], utility function $u_i : \mathbf{R}^n_+ \to \mathbf{R}_+$, which represents her preferences for the different bundles of goods, and an initial endowment of goods $w_i = (w_{i1}, \ldots, w_{in}) \in \mathbf{R}^n_+$. Let $W_j = \sum_i w_{ij}$ denote the total amount of good j in the market. The input size of M is defined to be the number of traders plus the number of goods plus the number of bits needed for encoding the rational numbers that describe the utility functions and initial endowments.

An equilibrium is a vector of prices $\pi = (\pi_1, \ldots, \pi_n) \in \mathbf{R}^n_+$ at which there is a bundle $\bar{x}_i = (\bar{x}_{i1}, \ldots, \bar{x}_{in}) \in \mathbf{R}^n_+$ of goods for each trader i such that the following two conditions hold: (i) For each trader i, the vector $\bar{x}_i$ maximizes $u_i(x)$ subject to the constraints $\pi^T x \leq \pi^T w_i$ and $x \in \mathbf{R}^n_+$, and (ii) For each good j, $\sum_i \bar{x}_{ij} \leq W_j$. Note that the constraint[4] $\pi^T x \leq \pi^T w_i$ in (i) says that the bundle x should cost no more than the *income* $\pi^T w_i$ of trader i.

For any price vector π, the vector $x_i(\pi)$ that maximizes $u_i(x)$ subject to the constraints $\pi^T x \leq \pi^T w_i$ and $x \in \mathbf{R}^n_+$ is called the *demand* [5] of trader i at prices π. The *excess demand* of trader i is $z_i(\pi) = x_i(\pi) - w_i$. Then $X_k(\pi) = \sum_i x_{ik}(\pi)$ denotes the *market demand* (or aggregate demand) of good k at prices π, and $Z_k(\pi) = X_k(\pi) - W_k = \sum_i z_{ik}(\pi)$ the *market excess demand* of good k at prices π. The vectors $X(\pi) = (X_1(\pi), \ldots, X_n(\pi))$ and $Z(\pi) = (Z_1(\pi), \ldots, Z_n(\pi))$ are called *market demand* (or aggregate demand) and *market excess demand*, respectively. The nonsatiability of the utility functions implies that at any price π for which the demand is well-defined, *Walras' Law* holds: $\pi^T Z(\pi) = 0$.

In terms of the excess demand function, the equilibrium is defined as a vector of prices $\pi = (\pi_1, \ldots, \pi_n) \in \mathbf{R}^n_+$ such that $Z_j(\pi) \leq 0$, for each j.

In this section we assume that (the excess demand of) the market M satisfies weak gross substitutability (WGS). That is, for any two sets of prices π and π' such that $0 < \pi_j \leq \pi'_j$, for each j, and $\pi_j < \pi'_j$ for some j, we have that $\pi_k = \pi'_k$ for any good k implies $Z_k(\pi) \leq Z_k(\pi')$. That is, increasing the prices for some of the goods while keeping some others fixed cannot cause a decrease in the aggregate demand for the goods whose price is fixed. Clearly, a market satisfies WGS if the excess demand of each individual trader does.

[3] u_i is nonsatiable if for any $x \in \mathbf{R}^n_+$ there is a $y \in \mathbf{R}^n_+$ such that $u_i(y) > u_i(x)$. Nonsatiation is considered, in the theory of equilibrium, a standard and extremely mild assumption (see [22], p. 42).

[4] Given two vectors x and y, we use $x \cdot y$ or $x^T y$ to denote their inner product.

[5] In the definitions we assume that the demand is a single-valued function of the prices, which is the case with most of the commonly used utility functions. The case of linear utility functions is an exception that can be easily handled in this framework [8].

To keep the definitions of approximate equilibria simple, we assume that all the utility functions $u()$ discussed in this paper satisfy $u(0) = 0$.

Definition 1 A bundle $x_i \in \mathbf{R}^n_+$ is a *μ-approximate demand*, for $\mu \geq 1$, of trader i at prices π if $u_i(x_i) \geq \frac{1}{\mu}u^*$ and $\pi^T x_i \leq \mu\pi^T w_i$, where $u^* = \max\{u_i(x) | x \in \mathbf{R}^n_+, \pi^T x \leq \pi^T w_i\}$.

A price vector $\pi \in \mathbf{R}^n_+$ is a *weak μ-approximate equilibrium* ($\mu \geq 1$) if there is a bundle x_i for each i such that (1) for each trader i, x_i is a μ-approximate demand of trader i at prices π, and (2) $\sum_i x_{ij} \leq \mu \sum_i w_{ij}$ for each good j.

In [8], it is shown that if π is a weak μ-approximate equilibrium, then there is a bundle $x'_i \in \mathbf{R}^n_+$ for each trader i such that (1) x'_i is a μ^2-approximate demand of trader i at prices π, (2) $\pi \cdot x'_i = \pi \cdot w_i$, and (3) for each good j, $\sum_i x'_{ij} = \sum_i w_{ij}$.

Definition 2 An exchange market M is said to be equipped with a *demand oracle* if there is an algorithm that takes as input a price vector $\pi \in \mathbf{Q}^n_+$ and a positive rational σ, and returns a vector $Y = (Y_1, Y_2, \ldots, Y_n)$ such that $|Y_j - Z_j(\pi)| \leq \sigma$ for all j. The algorithm is required to run in polynomial time in the input size and in $\log(1/\sigma)$.

We assume henceforth that the market M is equipped with a demand oracle.

Two Useful Transformations

We now describe a transformation that, given the exchange market M, produces a new market M' in which the total amount of each good is 1. The new utility function of the i-th trader is given by $u'_i(x_1, \ldots, x_n) = u_i(W_1x_1, \ldots, W_nx_n)$. It can be verified that, if $u_i()$ is concave, then $u'_i()$ is concave. The new initial endowment of the j-th good held by the i-th trader is $w'_{ij} = w_{ij}/W_j$. Let w'_i denote $(w'_{i1}, \ldots, w'_{in}) \in \mathbf{R}^n_+$. Clearly, $W'_j = \sum_i w'_{ij} = 1$.

The following lemma summarizes some key properties of the transformation.

LEMMA 3.
1. *For any $\mu \geq 1$, $(x_{i1}, \ldots, x_{in})$ is a μ-approximate demand at prices $(\pi_1, \ldots, \pi_n)$ for trader i in M' if and only if $(W_1x_{i1}, \ldots, W_nx_{in})$ is a μ-approximate demand at prices $(\frac{\pi_1}{W_1}, \ldots, \frac{\pi_n}{W_n})$ for trader i in M.*

2. *For any $\mu \geq 1$, $(\pi_1, \ldots, \pi_n)$ is a weak μ-approximate equilibrium for M' if and only if $(\frac{\pi_1}{W_1}, \ldots, \frac{\pi_n}{W_n})$ is a weak μ-approximate equilibrium for M.*

3. *M' has a demand oracle if M does. The excess demand of M' satisfies WGS if the excess demand of M does.*

We transform M' into another market $\hat{M}$ as follows. (This crucial transformation, which was discovered in the context of the present work on tâtonnement, also ended up simplifying the presentation in [8].) Let $0 < \eta \leq 1$ be a parameter. For each trader i, the new utility function and initial endowments are the same, that is, $\hat{u}_i() = u'_i()$, and $\hat{w}_i = w'_i$. The new market $\hat{M}$ has one extra trader, whose initial endowment is given by $\hat{w}_{m+1} = (\eta, \ldots, \eta)$, and whose utility function is the Cobb-Douglas[6] function $u_{m+1}(x_{m+1}) = \prod_j x_{m+1,j}^{1/n}$. A trader with this Cobb-Douglas utility function spends $1/n$-th of her budget on each good. Stated precisely, $\pi_j x_{m+1,j}(\pi) = \pi \cdot \hat{w}_{m+1}/n$. The extra trader allows us to show that at any equilibrium for $\hat{M}$ the ratio between the largest price and the smallest price is bounded above.

Note that the total amount of good j in the market $\hat{M}$ is $\hat{W}_j = \sum_{i=1}^{m+1} \hat{w}_{ij} = 1 + \eta$.

LEMMA 4. *(1) The market $\hat{M}$ has an equilibrium. (2) Every equilibrium π of $\hat{M}$ satisfies the condition $\frac{\max_j \pi_j}{\min_j \pi_j} \leq 2n/\eta$. (3) For any $\mu \geq 1$, a weak μ-approx equilibrium for $\hat{M}$ is a weak $\mu(1+\eta)$-approx equilibrium for M'. (4) $\hat{M}$ satisfies WGS if M' does. (5) $\hat{M}$ has a demand oracle if M' does.*

PROOF. (1) follows from standard arguments. Briefly, a quasi-equilibrium $\pi \in \mathbf{R}^n_+$ with $\sum_j \pi_j = 1$ always exists ([22], Chapter 17, Proposition 17.BB.2). At price π the income $\pi \cdot \hat{w}_{m+1}$ of the $(m+1)$'th trader is strictly positive. This ensures that that $\pi_j > 0$ for each good j. But this implies ([22], Chapter 17, Proposition 17.BB.1) that π is an equilibrium.

For (2), assume that the equilibrium price vector π is scaled so that $\max_j \pi_j = 1$. At price π, the income of the $(m+1)$'th trader is $\pi \cdot \hat{w}_{m+1} \geq \eta$. Since the $(m+1)$'th trader has the Cobb-Douglas utility function described above, she spends exactly a fraction $1/n$ of her income on each good. For any good k, her demand for the good is therefore at least $\frac{\eta}{n\pi_k}$. We must have $\frac{\eta}{n\pi_k} \leq \hat{W}_k = (1+\eta) \leq 2$, which implies that $\pi_k \geq \frac{\eta}{2n}$.

For (3), assume that π is a weak μ-approximate equilibrium for $\hat{M}$, and, for $1 \leq i \leq m+1$, let x_i be the corrresponding bundles. Evidently, for each $1 \leq i \leq m$, x_i is a μ-approximate demand for i in the market M', and thus also a $\mu(1+\eta)$-approximate demand. For each good k, we have $\sum_{i=1}^{m+1} x_{ik} \leq \mu\hat{W}_k$. Since $x_{m+1,k} \geq 0$, this implies that $\sum_{i=1}^{m} x_{ik} \leq \mu\hat{W}_k = \mu(1+\eta)W'_k$. Thus π is a weak $\mu(1+\eta)$-approx equilibrium for M'.

For (4), note that the individual excess demand of the $(m+1)$'th trader satisfies weak GS. The claim follows because the aggregate excess demand of $\hat{M}$ is the sum of the aggregate excess demand of M' and the individual excess demand of the $(m+1)$'th trader.

(5) follows for the same reason. □

We define $\Delta = \{\pi \in \mathbf{R}^n_+ | \eta/2n \leq \pi_j \leq 1 \text{ for each } j\}$. Note that Lemma 4 implies that $\hat{M}$ has an equilibrium price in Δ. We define $\Delta^+ = \{\pi \in \mathbf{R}^n_+ | \eta/2n - \eta/4n \leq \pi_j \leq 1 + \eta/4n \text{ for each } j\}$.

Abusing notation slightly, we henceforth let $Z(\pi)$ and $X(\pi)$ denote, respectively, the excess demand vector and the aggregate demand vector in the market $\hat{M}$.

LEMMA 5. *For any $\pi \in \Delta^+$, $||Z(\pi)||_2 \leq 8n^2/\eta$.*

PROOF. In the following sequence, the third inequality follows from Walras' Law using simple manipulations, the fourth inequality holds because $\pi \in \Delta^+$, and the fifth inequality holds because $\hat{W}_j \leq 2$ for each j.

[6]The Cobb-Douglas utility function has the general form $u_i(x) = \prod_j (x_{ij})^{a_{ij}}$, where $a_{ij} \geq 0$ and $\sum_j a_{ij} = 1$.

$$
\begin{aligned}
||Z(\pi)||_2 &\le \sum_j |Z_j(\pi)| \\
&\le \sum_j X_j(\pi) + \sum_j \hat{W}_j \\
&\le \frac{\max_k \pi_k}{\min_k \pi_k} \sum_j \hat{W}_j + \sum_j \hat{W}_j \\
&\le \frac{2n}{\eta} \sum_j \hat{W}_j + \sum_j \hat{W}_j \\
&\le \frac{4n^2}{\eta} + 2n \\
&\le \frac{8n^2}{\eta}.
\end{aligned}
$$

□

A Separation Result

Our strategy is to compute a $(1+\varepsilon)$-approximate equilibrium for $\hat{M}$. From Lemma 3 and Lemma 4 (applied with $\eta = \varepsilon$), this $(1+\varepsilon)$-approximate equilibrium will then be a $(1+O(\varepsilon))$-approximate equilibrium for M.

The following lemma says that if a vector $\pi \in \Delta^+$ is not a weak $(1+\varepsilon)$-approx equilibrium for $\hat{M}$, then the hyperplane normal to $Z(\pi)$ and passing through π separates π from all points within a distance δ of any equilibrium of $\hat{M}$ in Δ. The lower bound on δ is obtained by plugging in the parameters defining $\hat{M}$, Δ, and Δ^+ into the calculations of the proof of Lemma 3.2 in [8]. The key fact is that the two transformations described above ensure that these parameters are good enough.

LEMMA 6. *Let $\pi \in \Delta^+$ be a price vector that is not a weak $(1+\varepsilon)$-approximate equilibrium for $\hat{M}$, for some $\varepsilon > 0$. Then for any equilibrium $\hat{\pi} \in \Delta$, we have $\hat{\pi} \cdot Z(\pi) \ge \delta > 0$, where $1/\delta$ is bounded by a polynomial in n, $\frac{1}{\varepsilon}$, and $\frac{1}{\eta}$.*

The discrete tâtonnement process

Let π^0, the initial price, be any point in Δ. Suppose we have computed a sequence of prices $\pi^0, \ldots, \pi^{i-1}$. We compute π^i as follows. If $\pi^{i-1} \notin \Delta^+$, we let π^i be the point in Δ closest to π^{i-1}. In other words, $\pi^i_j = \pi^{i-1}_j$ if $\eta/2n \le \pi^{i-1}_j \le 1$; $\pi^i_j = 1$ if $\pi^{i-1}_j > 1$; $\pi^i_j = \eta/2n$ if $\pi^{i-1}_j < \eta/2n$.

If $\pi^{i-1} \in \Delta^+$, then we use the demand oracle to compute a vector $Y^{i-1} = (Y^{i-1}_1, \ldots, Y^{i-1}_n)$ such that $|Y^{i-1}_j - Z_j(\pi^{i-1})| \le \delta/4n$ for each j. We let

$$\pi^i = \pi^{i-1} + \frac{\delta}{2} \cdot \frac{1}{(9n^2/\eta)^2} Y^{i-1}.$$

Analysis

Let us fix an equilibrium π^* of $\hat{M}$ in Δ. We argue that in each iteration, the distance to π^* falls significantly so long as we don't encounter an approximate equilibrium in Δ^+. If $\pi^{i-1} \notin \Delta^+$, we have $|\pi^{i-1}_j - \pi^*_j| - |\pi^i_j - \pi^*_j| \ge 0$ for each j, while $|\pi^{i-1}_j - \pi^*_j| - |\pi^i_j - \pi^*_j| \ge \eta/4n$ for some j. From this it follows that

$$||\pi^* - \pi^{i-1}||^2 - ||\pi^* - \pi^i||^2 \ge (\eta/4n)^2.$$

Now suppose that $\pi^{i-1} \in \Delta^+$ and that π^{i-1} is not a weak $(1+\varepsilon)$-approx equilibrium for $\hat{M}$. By Lemma 6, $\pi^* \cdot Z(\pi^{i-1}) \ge \delta$. Since $\pi^{i-1} \cdot Z(\pi^{i-1}) = 0$ by Walras' Law, we have $(\pi^* - \pi^{i-1}) \cdot Z(\pi^{i-1}) \ge \delta$. Now

$$
\begin{aligned}
&(\pi^* - \pi^{i-1}) \cdot Y^{i-1} \\
&\ge (\pi^* - \pi^{i-1}) \cdot Z(\pi^{i-1}) \\
&\quad - \sum_j |Y^{i-1}_j - Z_j(\pi^{i-1})| * |\pi^*_j - \pi^{i-1}_j| \\
&\ge \delta - \sum_j \frac{\delta}{4n} \cdot 2 \ge \delta/2.
\end{aligned}
$$

Also note that since $||Z(\pi^{i-1})||_2 \le 8n^2/\eta$, we obtain, by a simple calculation, that $||Y^{i-1}||_2 \le 9n^2/\eta$.

Let q denote the vector $\frac{\delta}{2} \frac{1}{(9n^2/\eta)^2} Y^{i-1}$. We have

$$
\begin{aligned}
&(\pi^* - \pi^{i-1} - q) \cdot q \\
&= (\pi^* - \pi^{i-1}) \cdot q - q \cdot q \\
&= \frac{\delta}{2} \frac{1}{(9n^2/\eta)^2} ((\pi^* - \pi^{i-1}) \cdot Y^{i-1} \\
&\quad - \frac{\delta}{2} \frac{1}{(9n^2/\eta)^2} Y^{i-1} \cdot Y^{i-1}) \\
&\ge \frac{\delta}{2} \frac{1}{(9n^2/\eta)^2} (\delta/2 - \frac{\delta}{2} \frac{1}{(9n^2/\eta)^2} 9n^2/\eta) \ge 0.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&||\pi^* - \pi^{i-1}||^2 - ||\pi^* - \pi^i||^2 \\
&= ||\pi^* - \pi^{i-1}||^2 - ||\pi^* - \pi^{i-1} - q||^2 \\
&= (\pi^* - \pi^{i-1}) \cdot q + (\pi^* - \pi^{i-1} - q) \cdot q \\
&\ge (\pi^* - \pi^{i-1}) \cdot q \\
&= \frac{\delta}{2} \cdot \frac{1}{(9n^2/\eta)^2} (\pi^* - \pi^{i-1}) \cdot Y^{i-1} \\
&\ge \frac{\delta^2}{4(9n^2/\eta)^2},
\end{aligned}
$$

Suppose that every vector in the sequence $\pi^0, \ldots, \pi^k$ is either not in Δ^+ or is not a weak $(1+\varepsilon)$-approx equilibrium. We then have

$$
\begin{aligned}
&||\pi^* - \pi^{i-1}||^2 - ||\pi^* - \pi^i||^2 \\
&\ge \min\{\frac{\delta^2}{4(9n^2/\eta)^2}, (\eta/4n)^2\},
\end{aligned}
$$

for $1 \le i \le k$. Let μ denote $\min\{\frac{\delta^2}{4(9n^2/\eta)^2}, (\eta/4n)^2\}$. Adding these inequalities, we get

$$k\mu \le ||\pi^* - \pi^0||^2 - ||\pi^* - \pi^k||^2 \le n.$$

Thus, within n/μ iterations, the algorithm computes a price in Δ^+ which is a weak $(1+\varepsilon)$-approximate equilibrium for $\hat{M}$. It can be verified that the bound on μ is polynomial in the input size of the original market M, $1/\varepsilon$, and

$1/\eta$. Setting $\eta = \varepsilon$ in the transformation corresponding to Lemma 4, and putting everything together, we obtain:

THEOREM 7. *Let M be an exchange market whose excess demand function satisfies WGS, and suppose that M is equipped with a demand oracle. For any $\varepsilon > 0$, the tâtonnement based algorithm computes, in time polynomial in the input size of M and $1/\varepsilon$, a sequence of prices one of which is a weak $(1+\varepsilon)$-approx equilibrium for M.*

In order to actually pick the approximate equilibrium price from the sequence of prices, we need an efficient algorithm that recognizes an approximate equilibrium of M. In fact, it is sufficient for this algorithm to assert that a given price π is a weak $(1+2\varepsilon)$-approx equilibrium provided π is a weak $(1+\varepsilon)$-approx equilibrium. Since the problem of recognizing an approximate equilibrium is an explicitly presented convex programming problem, such an algorithm is generally quite easy to construct.

3. FISHER'S MODEL WITH MONOTONE DEMANDS

Here, we consider the Fisher's model, which is a market M of n goods desired by m utility maximizing buyers with fixed incomes. Each buyer has a concave, nonsatiable, utility function $u_i : \mathbf{R}^n_+ \to \mathbf{R}_+$ and an endowment $e_i > 0$ of *money*. There is a seller with an amount $q_j > 0$ of each good j. An equilibrium in the Fisher setting is a nonnegative vector of prices $\pi = (\pi_1, \ldots, \pi_n) \in \mathbf{R}^n_+$ at which there is a bundle $x_i = (x_{i1}, \ldots, x_{in}) \in \mathbf{R}^n_+$ of goods for each buyer i such that the following two conditions hold:

1. The vector x_i maximizes $u_i(x)$ subject to the constraints $\pi \cdot x \le e_i$ and $x \in \mathbf{R}^n_+$.
2. For each good j, $\sum_i x_{ij} = q_j$.

Walras' Law in this setting implies that at equilibrium, $\sum_j \pi_j q_j = \sum_i \sum_j \pi_j x_{ij} = \sum_i e_i$.

For any price vector π, the vector $x_i(\pi)$ that maximizes $u_i(x)$ subject to the constraints $\pi^T x \le e_i$ and $x \in \mathbf{R}^n_+$ is called the *demand* of buyer i at prices π. We continue to assume that the demand is a single-valued function of the price. The market demand $X(\pi)$ is defined to be $\sum_i x_i(\pi)$. The market excess demand $Z(\pi)$ is defined to be $X(\pi) - q$, where $q \in \mathbf{R}^n$ is the vector $(q_1, \ldots, q_n)$.

For $\mu \ge 1$, a μ-approx demand for trader i at prices π is an allocation $y_i \in \mathbf{R}^n_+$ such that $u_i(y_i) \ge u_i(x_i(\pi))/\mu$ and $\pi \cdot y_i \le \mu e_i$. We define a weak μ-approx equilibrium as a price π at which there are allocations x_i for each trader i such that (1) for each trader i, x_i is a μ-approx demand at prices π; (2) for each good j, $\sum_i x_{ij} \le \mu q_j$; and (3) $\sum_i e_i/\mu \le \sum_j \pi_j q_j \le \sum_i e_i$.

It is not hard to show that by scaling a μ-approx equilibrium π by the factor $\sum_i e_i / \sum_j \pi_j q_j$, we get a μ^2-approx equilibrium π' at which there are allocations y_i such that (1) for each trader i, y_i is a μ^2-approx demand at price π'; (2) for each good j, $\sum_i y_{ij} = q_j$; (3) $\pi \cdot y_i = e_i$ for each i; and (3) $\sum_i e_i = \sum_j \pi'_j q_j$.

The demand of the i-th trader is said to be *monotone* if for any pair of distinct positive prices $\pi, \pi' \in \mathbf{R}^n_+$, we have $(\pi - \pi') \cdot (x_i(\pi) - x_i(\pi')) \le 0$. If strict inequality always holds, then the demand is said to be *strictly monotone.* See [21] for an exposition of this property.

A Transformation. From the market M, we derive a new market $\hat{M}$. Let $0 < \eta \le 1$ be a parameter. For each trader i, the new utility function and money are the same, that is, $\hat{u}_i() = e_i()$, and $\hat{e}_i = e_i$. The new market $\hat{M}$ has one extra trader, whose income is given by $\hat{e}_{m+1} = \eta e$, where $e = \sum_{i=1}^m e_i$, and whose utility function is the Cobb-Douglas function $u_{m+1}(x_{m+1}) = \prod_j x_{m+1,j}^{1/n}$. The total amount of good j in the market $\hat{M}$ is $\hat{q}_j = (1+\eta) q_j$.

LEMMA 8. *(1) The market $\hat{M}$ has an equilibrium. (2) Every equilibrium π of $\hat{M}$ satisfies the condition $\eta e/nq_j \le \pi_j \le (1+\eta)e/q_j$. (3) For any $\mu \ge 1$, a weak μ-approx equilibrium for $\hat{M}$ is a weak $\mu(1+\eta)$-approx equilibrium for M. (5) $\hat{M}$ has a demand oracle if M does.*

Let $L = \min_j \eta e/nq_j$, and $U = \max_j (1+\eta)e/q_j$. We define the region $\Delta = \{\pi \in \mathbf{R}^n_+ | L \le \pi_j \le U$ for each $j\}$. Note that Lemma 8 implies that every equilibrium of $\hat{M}$ lies in Δ. We also define $\Delta^+ = \{\pi \in \mathbf{R}^n_+ | L - L/2 \le \pi_j \le U + L/2$ for each $j\}$.

Abusing notation slightly, we henceforth let $x_i(\pi)$, $X(\pi)$, and $Z(\pi)$ denote the individual demand, market demand, and market excess demand vectors in the market $\hat{M}$.

Separation. Our strategy is to set $\eta = \varepsilon$ in the above transformation and compute a weak $(1+\varepsilon)$-approximate equilibrium for $\hat{M}$. By Lemma 8, this will be a $(1+O(\varepsilon))$-approximate equilibrium for M. Let π^* henceforth denote an equilibrium for $\hat{M}$. Lemma 8 tells us that π^* is in Δ.

LEMMA 9. *Suppose that π is a vector such that $|\pi_j - \pi^*_j| \le \frac{\varepsilon}{3}\min\{\pi_j, \pi^*_j\}$ for each j, and*

$$\frac{1}{1+\varepsilon}\sum_i \hat{e}_i \le \sum_j \pi_j \hat{q}_j \le \sum_i \hat{e}_i.$$

Then π is a weak $(1+\varepsilon)$-approximate equilibrium for $\hat{M}$.

The following lemma says that if $\pi \in \Delta^+$ and $|\pi_j - \pi^*_j| > \frac{\varepsilon}{3}\min\{\pi_j, \pi^*_j\}$ for some j, then the hyperplane through π normal to $Z(\pi)$ separates π from all points within a large enough ball centered at π^*. The proof exploits the monotonicity of demand.

LEMMA 10. *Suppose that $\pi \in \Delta^+$ and that $|\pi_j - \pi^*_j| > \frac{\varepsilon}{3}\min\{\pi_j, \pi^*_j\}$ for some j. Then $(\pi^* - \pi)\cdot Z(\pi) \ge \delta$, where $\log 1/\delta$ and $\log ||Z(\pi)||_2$ are bounded by a polynomial in the input size.*

PROOF. The fact that $||Z(\pi)||_2$ is small enough follows easily from the fact that $\pi \in \Delta^+$.

Now suppose that

$$|\pi_k - \pi^*_k| > \frac{\varepsilon}{3}\min\{\pi_k, \pi^*_k\} \ge \frac{\varepsilon L}{6}.$$

Then

$$\begin{aligned}
&(\pi_k - \pi^*_k)(x_{m+1,k}(\pi) - x_{m+1,k}(\pi^*)) \\
&= (\pi_k - \pi^*_k)(\frac{\eta e}{n\pi_k} - \frac{\eta e}{n\pi^*_k}) \\
&= -\frac{\eta e}{n} \cdot \frac{(\pi_k - \pi^*_k)^2}{\pi_k \pi^*_k} \\
&\le -\frac{\eta e}{n}\frac{(\varepsilon L)^2}{6^2(U+L/2)^2} = -\delta
\end{aligned}$$

On the other hand, a similar argument for the other goods implies that for all $j \neq k$,

$$(\pi_j - \pi_j^*)(x_{m+1,j}(\pi) - x_{m+1,j}(\pi^*)) \leq 0.$$

We therefore have

$$(\pi - \pi^*) \cdot (x_{m+1}(\pi) - x_{m+1}(\pi^*)) \leq -\delta.$$

By monotonicity, we have that for each $1 \leq i \leq m$,

$$(\pi - \pi^*) \cdot (x_i(\pi) - x_i(\pi^*)) \leq 0.$$

Adding the inequalities over all $1 \leq i \leq m+1$, we get

$$(\pi - \pi^*) \cdot (X(\pi) - X(\pi^*)) \leq -\delta.$$

Since π^* is the equilibrium, $X(\pi^*) = \hat{q} = (\hat{q}_1, \ldots, \hat{q}_n)$, and thus $X(\pi) - X(\pi^*) = Z(\pi)$. We therefore have

$$(\pi - \pi^*) \cdot Z(\pi) \leq -\delta.$$

or

$$(\pi^* - \pi) \cdot Z(\pi) \geq \delta.$$

□

The Algorithm. The separation lemma leads to an algorithm, based on the Ellipsoid Method, for computing an approximate equilibrium in Δ^+. The separating hyperplane is computed as follows. If the current price π is not in Δ^+, then the separating hyperplane is any one of the violated constraints defining Δ^+. If the price π does not satisfy the two constraints

$$\frac{1}{1+\varepsilon} \sum_i \hat{e}_i \leq \sum_j \pi_j \hat{q}_j \leq \sum_i \hat{e}_i,$$

then the separating hyperplane is given by the constraint that is violated. Otherwise, the separating hyperplane, corresponding to Lemma 10, is the one normal to $Z(\pi)$ and passing through π. (Actually, the separating hyperplane will be normal to a close-enough approximation to $Z(\pi)$ that is returned by the demand oracle.)

Using Lemma 9 and Lemma 10, we can show that there is a $0 < \gamma < 1$, with $\log 1/\gamma$ bounded by a polynomial in the input size of M, $\log 1/\varepsilon$, and $\log 1/\eta$, such that the following condition holds: A half-ball of radius γ centered at π^* is contained in each of the ellipsoids produced by the Ellipsoid Algorithm so long as it does not generate a vector in Δ^+ that is a weak $(1+\varepsilon)$-approx equilibrium for $\hat{M}$. It therefore follows that the Ellipsoid Algorithm will generate a weak $(1+\varepsilon)$-approx equilibrium for $\hat{M}$ in time polynomial in the input size of M, $\log 1/\varepsilon$, and $\log 1/\eta$. Therefore we have established the main result of this section:

THEOREM 11. *There is an algorithm that takes as input (1) a description of a Fisher market M that is equipped with a demand oracle, and in which the individual demand of each buyer is monotone; (2) a parameter $\varepsilon > 0$, and returns a sequence of prices one of which is a weak $(1+\varepsilon)$-approx equilibrium for M. The running of time of the algorithm is polynomial in the size of M and in $\log 1/\varepsilon$.*

We suspect that the techniques in Section 2 can be used to obtain a simple polynomial-time approximation scheme for this problem based on tâtonnement.

4. A MODEL OF PRODUCTION

In this section, we consider a model M of an economy with m consumers, n goods, and τ firms. Like the trader in the model of exchange, each consumer i has a concave, nonsatiable, utility function $u_i : \mathbf{R}_+^n \to \mathbf{R}_+$, and an initial endowment of goods $w_i = (w_{i1}, \ldots, w_{in}) \in \mathbf{R}_+^n$. Each firm t has a set of production possibilities S_t, which we assume to be a closed, bounded, subset of $\mathbf{R}_+^n$. We may assume that each set S_t is input to the algorithm as (the set of solutions to) a system of linear inequalities. Each consumer i has a *share* $\theta_{it} \geq 0$ in the t-th firm. For each firm t, we have $\sum_{i=1}^m \theta_{it} = 1$.

An equilibrium is a vector of prices $\pi = (\pi_1, \ldots, \pi_n) \in \mathbf{R}_+^n$ at which there is a bundle $x_i = (x_{i1}, \ldots, x_{in}) \in \mathbf{R}_+^n$ of goods for each trader i and a bundle $y_t = (y_{t1}, \ldots, y_{tn}) \in S_t$ for each firm t such that the following three conditions hold: (i) For each firm t, the vector y_t maximizes the *profit* $\pi \cdot y$ over all $y \in S_t$. (ii) For each consumer i, the vector x_i maximizes $u_i(x)$ subject to the constraints $\pi \cdot x \leq \pi \cdot w_i + \sum_t \theta_{it} \pi \cdot y_t$ and $x \in \mathbf{R}_+^n$, and (iii) For each good j, $\sum_i x_{ij} \leq \sum_t y_{tj} + \sum_i w_{ij}$.

For any price vector π, the set of vectors $y_t(\pi)$ that maximize $\pi \cdot y$ subject to $y \in S_t$ is called the *supply* of firm t at price π. Note that the supply is a set-valued function of the price, that is, a *correspondence*. The *profit* function of the firm t is defined by $\mathrm{Pr}_t(\pi) = \pi \cdot y$ for any $y \in y_t(\pi)$. We let $Y(\pi) = \sum_i w_i + \sum_t y_t(\pi)$ denote the *aggregate supply* correspondence.

For any price vector π, the vector $x_i(\pi)$ that maximizes $u_i(x)$ subject to the constraints $\pi^T x \leq \pi^T w_i + \sum_t \theta_{it} \mathrm{Pr}_t(\pi)$ and $x \in \mathbf{R}_+^n$ is called the *demand* of trader i at price π. We continue to assume that the demand is a single valued function of the price. We let $X(\pi) = \sum_i x_i(\pi)$ denote the *aggregate demand* function. The *aggregate excess demand* correspondence is given by $Z(\pi) = X(\pi) - Y(\pi)$. At any price $\pi \in \mathbf{R}_+^n$ where $Z(\pi)$ is well-defined, Walras' Law holds: $\pi \cdot z = 0$ for any $z \in Z(\pi)$.

Note that an equilibrium is simply a vector $\pi \in \mathbf{R}_+^n$ at which there is a vector $z \in Z(\pi)$ such that $z_j \leq 0$ for each j.

For any $\mu \geq 1$, we define a *weak* μ-approx equilibrium to be a price $\pi \in \mathbf{R}_+^n$ at which there are allocations $x_i' \in \mathbf{R}_+^n$ for each i and production plans $y_t' \in S_t$ for each t such that (i) for each i, x_i' is a μ-*approx demand* of trader i at price π, that is, $u_i(x_i') \geq u_i(x_i(\pi))/\mu$ and $\pi \cdot x_i' \leq \mu(\pi \cdot w_i + \sum_t \theta_{it} \mathrm{Pr}_t(\pi))$; (ii) for each firm t, y_t' almost maximizes the profit at price π, that is, $\pi \cdot y_t' \geq \mathrm{Pr}_t(\pi)/\mu$; (iii) $\sum_i x_i' \leq \sum_i w_i + \sum_t y_t'$. It can be shown that if π is a μ-approx equilibrium, we can compute a possibly different set of allocations x_i'' for each i such that

1. $u_i(x_i'') \geq u_i(x_i(\pi))/\mu^2$ for each i ;
2. $\pi \cdot x_i'' = \pi \cdot w_i + \sum_t \theta_{it} \pi \cdot y_t' \leq \pi \cdot w_i + \sum_t \theta_{it} \mathrm{Pr}_t(\pi)$;
3. $\sum_i x_i'' = \sum_i w_i + \sum_t y_t'$.

We now describe the gross substitutability assumption that is made in this section. For any $w \in \mathbf{R}_+^n$, we define a "demand" function for the ith consumer $f_i^w()$. At any price π, let $f_i^w(\pi)$ denote the vector in $\mathbf{R}_+^n$ that maximizes $u_i(x)$ subject to the constraint $\pi \cdot x \leq \pi \cdot w$. Note that $f_i^w()$ corresponds to the demand of a trader in an *exchange* model with utility function u_i and initial endowment w. Our assumption is that such a demand must satisfy WGS, that is, for any two prices π and π' such that $\pi_j \leq \pi'_j$ for each j, $\pi_k = \pi'_k$ for some k implies $f_{ik}^w(\pi') \geq f_{ik}^w(\pi)$.

We will also assume that the demands of each consumer are *normal*. That is, if at any price we increase the income of a consumer without changing the prices of the goods, then the demand of the consumer for any good does not decrease. This assumption is satisfied by most of the utility functions leading to gross substitutability (and certainly by homogeneous functions or separable functions).

We assume that each good j is either present in positive amount in the initial endowment of some consumer or that there is some firm that can produce a positive amount of it. That is, either $w_{ij} > 0$ for some i or $y_{tj} > 0$ for some $y_t \in S_t$. Note that our assumption on the boundedness of the production sets implies that there exists $U > 0$ with $\log U$ polynomial in the input size such that for any good j, we have $\sum_i w_{ij} + \sum_t y_{tj} \leq U$, where $y_t \in S_t$.

A Useful Transformation

We transform M into another market $\hat{M}$ as follows. Let $0 < \eta \leq 1$ be a parameter. For each consumer i, the new utility function and initial endowments are the same, that is, $\hat{u}_i() = u_i()$, and $\hat{w}_i = w_i$. The new market $\hat{M}$ has one extra consumer, whose initial endowment of the j-th good is given by $\hat{w}_{m+1,j} = \eta W_j = \eta \sum_i w_{ij}$, and whose utility function is the Cobb-Douglas function $u_{m+1}(x_{m+1}) = \prod_j x_{m+1,j}^{1/n}$.

For each firm t, the new production set is given by $\hat{S}_t = (1+\eta)S_t = \{(1+\eta)y \mid y \in S_t\}$. For each $1 \leq i \leq m$, the new share of i-th trader in the t-th firm is $\hat{\theta}_{it} = \frac{1}{1+\eta}\theta_{it}$. The share of the $(m+1)$-th trader in the t-th firm is $\hat{\theta}_{m+1,t} = \frac{\eta}{1+\eta}$.

LEMMA 12. *(1) The market $\hat{M}$ has an equilibrium. (2) At every equilibrium π of $\hat{M}$, we have $\log \frac{\max_j \pi_j}{\min_j \pi_j} \leq U_1$, where U_1 is bounded by a polynomial in the input size and $\log 1/\eta$. (3) At any equilibrium π of $\hat{M}$, the aggregate supply $Y_j(\pi)$ of any good π satisfies $2^{-L_2} \leq Y_j(\pi) \leq 2^{U_2}$, where $U_2, L_2 > 0$ are bounded above by a polynomial in the input size and $\log 1/\eta$. (3) For any $\mu \geq 1$, a weak μ-approx equilibrium for $\hat{M}$ is a weak $\mu(1+\eta)$-approx equilibrium for M. (4) $\hat{M}$ has a demand oracle if M does.*

Remark : For part (4), the demand oracle for M needs to return an approximation to the aggregate demand in M as well as an approximation to some point in the aggregate supply in M. It is not sufficient to merely return an approximation to some point in the aggregate excess demand. We assume this stronger oracle in this section.

We define $\Delta = \{\pi \in \mathbf{R}_+^n | 2^{-U_1} \leq \pi_j \leq 1 \text{ for each } j\}$. Note that the lemma implies that $\hat{M}$ has an equilibrium price in Δ. We define $\Delta^+ = \{\pi \in \mathbf{R}_+^n | 2^{-U_1} - 2^{-U_1}/2 \leq \pi_j \leq 1 + 2^{-U_1}/2 \text{ for each } j\}$.

Abusing notation slightly, we henceforth let $x_i(\pi)$, $Z(\pi)$, etc. denote the functions/correspondences in the market $\hat{M}$.

A Separation Result

Our strategy is to compute a $(1+\varepsilon)$-approximate equilibrium for $\hat{M}$. From Lemma 12 (applied with η set to ε), this will then be a $(1 + O(\varepsilon))$-approximate equilibrium for M.

From now on, let $\pi^* \in \Delta$ denote an equilibrium for $\hat{M}$. The following lemma says that if $\pi \in \mathbf{R}_+^n$ is "close" to π^*, then π is an approximate equilibrium.

LEMMA 13. *Let $\pi \in \mathbf{R}_+^n$ be a price vector such that*

$$\max_j \frac{\pi_j}{\pi_j^*} - \min_j \frac{\pi_j}{\pi_j^*} \leq \frac{\varepsilon}{3} \min_j \frac{\pi_j}{\pi_j^*},$$

where $0 < \varepsilon < 1$. Then π is a weak $(1+\varepsilon)$-approx equilibrium for $\hat{M}$.

PROOF. By scaling π, we may assume that $\min_j \frac{\pi_j}{\pi_j^*} = 1$. Then the hypothesis of the lemma tells us that $\pi_j^* \leq \pi_j \leq (1+\varepsilon/3)\pi_j^*$. From this, it can be shown that for each firm t, the equilibrium production plan y_t^* approximately maximizes the profit at π. Also for each consumer i, the equilibrium allocation x_i^* is an approximate demand at π. Since we also have $\sum_i x_i^* \leq \sum_i \hat{w}_i + \sum_t y_t^*$, we can conclude that π is a weak $(1+\varepsilon)$-approx equilibrium. □

The following lemma says that if a vector $\pi \in \Delta^+$ is not close to π^* in the sense of Lemma 13, then the hyperplane normal to $Z(\pi)$ and passing through π separates π from all points within a distance $\delta > 0$ of π^*. The proof of this lemma weaves together a "reduction" due to Primak [28] of production to exchange and the separation argument of a lemma in [8].

LEMMA 14. *Let $\pi \in \Delta^+$ be a price vector such that*

$$\max_j \frac{\pi_j}{\pi_j^*} - \min_j \frac{\pi_j}{\pi_j^*} > \frac{\varepsilon}{3} \min_j \frac{\pi_j}{\pi_j^*},$$

for some $\varepsilon > 0$. Then for any $z \in Z(\pi)$, we have $\pi^* \cdot z \geq \delta$, where $\delta \geq 1/2^{E_1}$, and E_1 is bounded by a polynomial in the input size of M, $\log 1/\eta$, and $\log\frac{1}{\varepsilon}$. Moreover $||z||_2 \leq 2^{E_2}$, where E_2 is bounded by a polynomial in the input size of M and $\log 1/\eta$. (Note that $\pi \cdot z = 0$ by Walras' Law.)

PROOF. Recall that $x_i(\pi^*)$ denotes the demand of the i-th consumer at π^*. For $i \leq t \leq \tau$, let $y_t^* \in y_t(\pi^*)$ denote a profit maximizing plan for the t-th producer which ensures that $\sum_i x_i(\pi^*) \leq \sum_i \hat{w}_i + \sum_t y_t(\pi^*)$. We construct an exchange economy M' in which, for each $1 \leq i \leq m+1$, the i-th trader has the utility function $\hat{u}_i$ and the initial endowment $w_i + \sum_t \hat{\theta}_{it} y_t^*$. It is easy to verify that π^* is also an equilibrium of the exchange economy M'. Note that the upper and lower bounds of Lemma 12 (3) apply to the total initial endowment of each good in M'.

Let $Z'()$ denote the excess demand function of M'. Note that $Z'()$ is a single-valued function. We now apply the argument of Lemma 3.2 from [8], where the exchange economy that is considered has the same form as M'.[7] A careful analysis of the steps in [8] implies that $\pi^* \cdot Z'(\pi) \geq \delta$. The bound

[7]The premise of Lemma 3.2 of [8] that π is not a weak $(1+\varepsilon)$-approx equilibrium is only used to show that $\max_j \frac{\pi_j}{\pi_j^*} - \min_j \frac{\pi_j}{\pi_j^*} > \frac{\varepsilon}{3} \min_j \frac{\pi_j}{\pi_j^*}$.

$\delta \geq 1/2^{E_1}$ follows by plugging in the bounds on the initial endowments in M' implied by Lemma 12 (3). Note that gross substitutability is used in showing that $\pi^* \cdot Z'(\pi) \geq \delta$.

We now argue that $\pi^* \cdot z \geq \pi^* \cdot Z'(\pi)$. Let us write $Z'(\pi) = \sum_i x'_i(\pi) - \sum_i w_i - \sum_i \sum_t \hat{\theta}_{it} y^*_t$, where $x'_i(\pi)$ is the demand of trader i in M'. Let $z = \sum_i x_i(\pi) - \sum_i w_i - \sum_t y_t$, where $y_t \in y_t(\pi)$. Since $\pi \cdot y^*_t \leq \pi \cdot y_t$, we have

$$\pi \cdot w_i + \sum_t \hat{\theta}_{it} \pi \cdot y_t \geq \pi \cdot w_i + \sum_t \hat{\theta}_{it} \pi \cdot y^*_t.$$

That is, the income of the i-th trader in $\hat{M}$ is greater than or equal to her income in M'. The assumption that the demand of i is normal then implies that $x_{ij}(\pi) \geq x'_{ij}(\pi)$ for each good j, which implies that $\pi^* \cdot x_i(\pi) \geq \pi^* \cdot x'_i(\pi)$.

On the other hand we have $\pi^* \cdot y^*_t \geq \pi^* \cdot y_t$, since y^*_t is profit maximizing at π^*. Thus, we have

$$\begin{aligned}
\pi^* \cdot z &= \sum_i \pi^* \cdot x_i(\pi) - \sum_i \pi^* \cdot w_i - \sum_t \pi^* \cdot y_t \\
&\geq \sum_i \pi^* \cdot x'_i(\pi) - \sum_i \pi^* \cdot w_i - \sum_t \pi^* \cdot y^*_t \\
&= \pi^* \cdot \left(\sum_i x'_i(\pi) \right) - \pi^* \cdot \left(\sum_i w_i \right) \\
&\quad -\pi^* \cdot \left(\sum_i \sum_t \hat{\theta}_{it} y^*_t \right) \\
&= \pi^* \cdot Z'(\pi) \geq \delta.
\end{aligned}$$

The claimed bound on $||z||_2$ follows easily from the fact that $\pi \in \Delta^+$ and the upper bound in Lemma 12 (3). $\square$

The Algorithm.

Lemmas 13 and 14 lead to an algorithm, based on the Ellipsoid Method, for computing a $(1+\varepsilon)$-approx equilibrium of $\hat{M}$ that lies in Δ^+. The running time of this algorithm is polynomial in the input size of M, $\log 1/\eta$ and $\log 1/\varepsilon$. Briefly, the initial ellipsoid is a ball containing Δ^+. If the center of the current ellipsoid does not lie in Δ^+, the separating hyperplane is the constraint defining Δ^+ that is violated. If the center π of the current ellipsoid lies in Δ^+, then the separating hyperplane is the one normal to $Z(\pi)$. (Actually, an approximation to $Z(\pi)$ computed using the demand oracle.) The argument for the correctness of this algorithm is very similar to the one sketched for the Fisher case in Section 3. We conclude with the main result of this section.

Theorem 15. *There is an algorithm that takes as input (1) a description of a production economy M that is equipped with a demand oracle, and that satisfies the gross substitutability and the normality assumptions; and (2) a parameter $\varepsilon > 0$, and returns a sequence of prices one of which is a weak $(1+\varepsilon)$-approx equilibrium for M. The running of time of the algorithm is polynomial in the input size of M and* $\log 1/\varepsilon$.

5. CONCLUSIONS

It is possible, using the techniques in Section 4, to extend the results in Section 3 to a model where instead of a seller there is production. This is the model considered in [30, 19], for which we now obtain polynomial-time algorithms for buyers with monotone demands. Details will appear in the full version.

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