Algorithms (CS:3330:0001 or 22C:031:001)
Homework 8

The problems in this homework are on polynomial time reducibility and NP-completeness. The homework is due in class on Friday, December 12. If you can't make it to class, drop it in my mailbox in the MacLean Hall mailroom.

1. An undirected graph $G = (V, E)$ is said to be three-colorable if there is an assignment $\chi : V \to \{1, 2, 3\}$ such that for each edge $(u, v) \in E$, $\chi(u) \neq \chi(v)$. In other words, the graph is three-colorable if we can assign one of three colors to each vertex in such a way that the endpoints of any edge get different colors. In the algorithmic problem we call Three-Colorability, the input is an undirected graph G and the goal is to determine if G is three-colorable. Show that Three-Colorability is polynomial time reducible to CNF-Satisfiability by describing the reduction.

   Hint: This is very similar to a reduction we did in class from Two-Colorability to CNF-Satisfiability. For each vertex v, introduce three boolean variables x_v^1, x_v^2, x_v^3. The interpretation is that we want the assignment $x_v^i = 1$ to correspond to $\chi(v) = i$, that is, v being colored i. Add clauses to enforce this interpretation and to enforce the requirement that the endpoints of any edge should not get the same color. (3 points)

2. A store trying to analyze the behavior of its customers will often maintain a two-dimensioanl array A, where the rows correspond to its customers and the columns correspond to the products it sells. The entry $A[i, j]$ specifies the quantity of product j that has been purchased by customer i.

   Here's a tiny example of such an array A.

	liquid detergent	beer	diapers	cat litter
Raj	0	6	0	3
Alanis	2	3	0	0
Chelsea	0	0	0	7

   One thing that a store might want to do with this data is the following. Let us say that a subset S of the customers is *diverse* if no two of the customers in S have ever bought the same product (i.e., for each product, at most one of the customers in S has ever bought it). A diverse set of customers can be useful, for example, as a target pool for market research.

   We can now define the Diverse Subset Problem as follows: Given an $m \times n$ array A as defined above, and a number $k \leq m$, is there a subset of at least k customers that is diverse?

   Show that Diverse Subset is NP-complete (3.5 points). This problem is Exercise 2 in Chapter 8 of the text.

3. Consider an instance of the Satisfiability problem, specified by clauses $C_1, \ldots, C_k$ over a set of Boolean variables $x_1, \ldots, x_n$. We say that the instance is *monotone* if each literal in each clause consists of a nonnegated variable; that is, each term is equal to x_i, for some i, rather than $\bar{x}_i$. Monotone instances of Satisfiability are very easy to solve: They are always satisfiable, by setting each variable to 1.

   For example, suppose we have the three clauses

   $$(x_1 \vee x_2), (x_1 \vee x_3), (x_2 \vee x_3).$$

   This is monotone, and indeed the assignment that sets all three variables to 1 satisfies all the clauses. But we can observe that this is not the only satisfying assignment; we could also have set x_1 and x_2 to 1, and x_3 to 0. Indeed, for any monotone instance, it is natural to ask how few variables we need to set to 1 in order to satisfy it.

   Given a monotone instance of Satisfiability, together with a number k, the problem of *Monotone Satisfiability with Few True Variables* asks: Is there a satisfying assignment for the instance in which at most k variables are set to 1? Prove that this problem is NP-complete. (3.5 Points) This is Exercise 6 in Chapter 8 of the text.

Let's review some policy from the first day handout. "For homework problems, collaboration is allowed, assuming each of you has first spent some time (about 30 minutes) working on the problem yourself. However, no written transcript (electronic or otherwise) of the collaborative discussion should be taken from the discussion by any participant. Furthermore, discussing ideas is okay but viewing solutions of others is not. It will be assumed that each of you is capable of orally explaining the solution that you turn in, so do not turn in something you don't understand. Students are responsible for understanding this policy; if you have questions, ask for clarification."

Please note that although the correctness of your solution counts for a significant fraction of your score, the quality of your writing and a demonstration that you understood the question and made a serious attempt at solving it also count for a significant fraction.