Algorithms (CS:3330:0001 or 22C:031:001)
Homework 6

The problems in this homework are exercises in recursive thinking (of the type that occurs in developing dynamic programming algorithms). In particular, each exercise asks you to write a recurrence. You don't need to justify your recurrence, though you are certainly welcome to. However, you should ideally have a justification in mind for each recurrence.

The homework is due in class on Monday, November 10. If you can't make it to class, drop it in my mailbox in the MacLean Hall mailroom.

1. Let $X[1..n]$ be a sequence of n integers. A *subsequence* of a sequence like $X[1..n]$ is any sequence obtained by dropping certain elements from the sequence. For example, if $X[1..6] = \langle 25, 7, 16, 8, 12, 40 \rangle$, then $\langle 25, 16, 40 \rangle$, $\langle 25, 40 \rangle$, and $\langle 7, 8, 12, 40 \rangle$ are some of its subsequences. A subsequence is said to be *increasing* if each element in it is greater than the previous element. Thus, $\langle 25, 40 \rangle$, and $\langle 7, 8, 12, 40 \rangle$ are increasing subsequences of $X[1..6]$, whereas $\langle 25, 16, 40 \rangle$ is not an increasing subsequence.

   Let $\lambda(i)$ denote the *length* of the longest increasing subsequence of $X[i..n]$ among all increasing subsequences that begin with $X[i]$. Thus, in the example, $\lambda(1) = 2$, $\lambda(2) = 4$, $\lambda(3) = 2$, $\lambda(4) = 3$, $\lambda(5) = 2$, and $\lambda(6) = 1$.

   Suppose that $1 \le i \le n - 1$. Express $\lambda(i)$ in terms of $\lambda(i+1), \lambda(i+2), \ldots, \lambda(n)$. (Your answer should allow us to easily compute $\lambda(i)$ from $\lambda(i+1), \lambda(i+2), \ldots, \lambda(n)$ and the input sequence.) (3 points)

2. Let $A[1..m]$ and $B[1..n]$ be two sequences of integers. A *common supersequence* of A and B is a sequence for which both $A[1..m]$ and $B[1..n]$ are subsequences. For example, if $A[1..3] = \langle 31, 12, 45 \rangle$ and $B[1..3] = \langle 16, 12, 13 \rangle$, then $\langle 31, 12, 45, 16, 12, 13 \rangle$ and $\langle 16, 31, 12, 45, 13 \rangle$ are both common supersequences of $A[1..3]$ and $B[1..3]$.

   For $0 \le i \le m$ and $0 \le j \le n$, let $\mathrm{scs}(i, j)$ denote the *length* of the shortest common supersequence of $A[1..i]$ and $B[1..j]$. In the above example $\mathrm{scs}(3, 3) = 5$, $\mathrm{scs}(2, 2) = 3$, and $\mathrm{scs}(1, 2) = 3$.

   Supoose that $1 \le i \le m$ and $1 \le j \le n$. Express $\mathrm{scs}(i, j)$ in terms of $\mathrm{scs}(i-1, j-1)$, $\mathrm{scs}(i, j-1)$, and $\mathrm{scs}(i-1, j)$. (Your answer should allow us to easily compute $\mathrm{scs}(i, j)$ from $\mathrm{scs}(i-1, j-1)$, $\mathrm{scs}(i, j-1)$, $\mathrm{scs}(i-1, j)$, and the two input sequences.) (4 points)

3. Suppose that the vertices of a directed graph $G = (V, E)$ are $\{1, 2, \ldots, n\}$ and every edge (i, j) has $i < j$. That is, we are given a topological ordering of a directed acyclic graph. Let $\mu(j)$ denote the *number* of paths in the graph G from vertex 1 to vertex j. See the figure for an example.

   Suppose $2 \le j \le n$. Express $\mu(j)$ in terms of $\mu(1), \mu(2), \ldots, \mu(j-1)$. (Your answer should allow us to easily compute $\mu(j)$ from $\mu(1), \mu(2), \ldots, \mu(j-1)$ and the input graph.) (3 points)

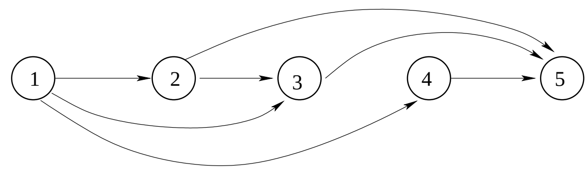

Figure 1: $\mu(1) = 1$, $\mu(2) = 1$, $\mu(3) = 2$, $\mu(4) = 1$, $\mu(5) = 4$.

Let's review some policy from the first day handout. "For homework problems, collaboration is allowed, assuming each of you has first spent some time (about 30 minutes) working on the problem yourself. However, no written transcript (electronic or otherwise) of the collaborative discussion should be taken from the discussion by any participant. Furthermore, discussing ideas is okay but viewing solutions of others is not. It will be assumed that each of you is capable of orally explaining the solution that you turn in, so do not turn in something you don't understand. Students are responsible for understanding this policy; if you have questions, ask for clarification."

Please note that although the correctness of your solution counts for a significant fraction of your score, the quality of your writing and a demonstration that you understood the question and made a serious attempt at solving it also count for a significant fraction.