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Spectral sets whose spectrum is a lattice with a base. (English. English summary)

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Spectral sets and factorizations of finite abelian groups. (English. English summary)

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FEATURED REVIEW.

Both of these papers deal with a special case of a long-standing conjecture of B. Fuglede concerning tilings of $\mathbf{R}^n$ and basis properties of $L^2(\Omega)$, where Ω is some subset in $\mathbf{R}^n$ of finite positive Lebesgue measure. The papers also both acknowledge suggestions from the reviewer. Fuglede conjectured [J. Functional Analysis **16** (1974), 101–121; MR **57** #10500] that a subset $\Omega \subset \mathbf{R}^n$, of finite positive Lebesgue measure, tiles $\mathbf{R}^n$ by translations by some subset $S \subset \mathbf{R}^n$ if and only if $L^2(\Omega)$ has an orthogonal basis of the form $\{e^{i\lambda\cdot x}\colon\ \lambda \in \Lambda\}$ for some subset $\Lambda \subset \mathbf{R}^n$. (The functions $e_\lambda(x) = e^{i\lambda\cdot x}$ are restricted to Ω and then Λ is said to be a spectrum.) The tiling is up to measure zero, i.e., it is assumed that the distinct translates $s + \Omega$ and $s' + \Omega$ for $s, s' \in S$ overlap on a set of at most Lebesgue measure zero.

To give the basis for the conjecture, we provide some history: Fuglede's paper resulted from a question raised much earlier (1957) by I. E. Segal, who in turn (later) suggested to the reviewer that the origin is from the desire to understand the candidates for spectral transforms of the partial derivatives $\partial/\partial x_j$, $j = 1, \cdots, n$, on bounded open sets $\Omega \subset \mathbf{R}^n$, going back to von Neumann. Segal asked which Ω admit commuting selfadjoint extension operators H_j of the partial derivatives $(1/\sqrt{-1})\partial/\partial x_j$ on the dense subspace $C_c^\infty(\Omega)$ in $L^2(\Omega)$. If commuting extensions exist, we say that Ω has the extension property. In that case, we have the identity $\sqrt{-1}H_j f = (\partial/\partial x_j)f$, $\forall f \in C_c^\infty(\Omega)$, $\forall j = 1, \cdots, n$. But Fuglede also showed that this property is very restrictive: the triangle and the disk in $\mathbf{R}^2$ do not have the extension property. Fuglede further showed that the extension property is closely connected to the spectral property, i.e., the presence of a subset $\Lambda \subset \mathbf{R}^n$ such that the exponentials $\{e^{i\lambda\cdot x}\colon\ \lambda \in \Lambda\}$ form an orthogonal basis for $L^2(\Omega)$. If that is the case, then commuting selfadjoint extensions

H_j may be defined by $H_j e_{(\lambda_1,\cdots,\lambda_n)} = \lambda_j e_{(\lambda_1,\cdots,\lambda_n)}$, $j = 1, \cdots, n$, where $e_\lambda := e^{i\lambda\cdot x}$, $x \in \Omega$. If Ω is further assumed to be connected, the converse also holds. (This involved earlier work of Fuglede, Pedersen, and the reviewer (1982).) The original motivation and justification for the conjecture came from the von Neumann-Stone spectral theorem. To see this, note that if Ω has the extension property then we may define a unitary representation U of the group $\mathbf{R}^n$, acting on the Hilbert space $L^2(\Omega)$, via $U(t_1, \cdots, t_n) = \exp(i \sum_{j=1}^n t_j H_j)$, $t \in \mathbf{R}^n$; and there is a projection-valued measure E on $\mathbf{R}^n$ with values in projections on $L^2(\Omega)$ such that $U(t) = \int_{\mathbf{R}^n} e^{i\lambda\cdot t} E(d\lambda)$. The support of E is the spectrum Λ of U, and the above-mentioned result states that this spectrum Λ must be discrete, the spectral measure E must be atomic, and $E(\{\lambda\})$ must be the one-dimensional projection onto $\mathbf{C}e_\lambda$ if $\lambda \in \Lambda$, and zero if not.

While the spectral theorem had been used this way earlier for one dimension, there was, prior to Fuglede's paper, very little, or nothing, for higher dimensions. In making the connection between (i) the extension property, (ii) the spectral property, and (iii) the tile property for some subset $\Omega \subset \mathbf{R}^n$ (as specified), Fuglede also noticed that (ii)–(iii) make sense for other groups, and are both interesting and nontrivial for finite abelian groups, and even that the problem for the finite cyclic groups $\mathbf{Z}_k = \mathbf{Z}/k\mathbf{Z}$ is closely connected to the problem for $\mathbf{R}^n$; and he stated two results (without proofs) to that effect. The proofs have now been given by Lagarias and Wang in the second paper under review.

It is not immediately clear to the uninitiated that there is any even intuitive connection between properties (ii) and (iii) for general sets $\Omega \subset \mathbf{R}^n$. To make this connection (the conjecture is that they are the same!), note that, in one dimension, the selfadjoint extensions H of $(1/i)\frac{d}{dx}$ on $C_c^\infty(0,1)$ in $L^2(0,1)$ are parameterized by $\mathbf{T}$; specifically, any extension H must have spectrum of the form $a + \mathbf{Z}$ for some $a \in [0,1\rangle$; in fact, a given $H = H_a$ is specified by its domain $\mathcal{D}(H_a) = \{f \in C^1(0,1)\colon\ f(0) = e^{i2\pi a} f(1)\}$, and moreover, the unitary one-parameter group $U_a(t) = e^{itH_a}$ is induced in the sense of Mackey by the one-dimensional representation $n \mapsto e^{i2\pi a\cdot n}$ of $\mathbf{Z}$. The reviewer [Adv. in Math. **44** (1982), no. 2, 105–120; MR 84k:47024] showed in one dimension that the unitary representation U of $\mathbf{R}$ on $L^2(\Omega)$ arising from a general domain $\Omega \subset \mathbf{R}$ with the extension property must be induced (in Mackey's sense) not just from a character of some lattice Γ in $\mathbf{R}$, but from a representation of Γ which is possibly of higher (finite) dimension. He had an analogous result for $\mathbf{R}^n$, when $n > 1$, but then the rank of Γ could be smaller than n. However, geometric conditions were given, in the 1982 paper, under which $\mathrm{rank}(\Gamma) = n$. But those were restricting assumptions, and the general case for $n > 1$ is to this day not well understood. If $\mathrm{rank}(\Gamma) = n$, then the spectrum of the corresponding commuting tuple $(H_1, \cdots, H_n)$ of selfadjoint operators must have the form $A + \Gamma^\circ = \{a + \xi\colon\ a \in A,\ \xi \in \Gamma^\circ\}$, where Γ° is the dual lattice and A is a finite subset of $\mathbf{R}^n$ such that distinct points in A correspond to different points in the torus $\mathbf{R}^n/\Gamma^\circ \simeq (\Gamma)\widehat{}$ (the compact dual group), under the quotient morphism $\mathbf{R}^n \to \mathbf{R}^n/\Gamma^\circ$. After changing basis in $\mathbf{R}^n$, we note that sets in $\mathbf{R}^n$ of the form $A + \mathbf{Z}^n$ are natural candidates for Λ (spectrum) when Ω has the spectral property with spectrum Λ, and when Ω is a tile with translation set S, i.e., the candidates for translation vectors would have the same form.

While the nature of more general examples of (ii)–(iii) is not known, this setting of quasiperiodicity is reasonable, and it is adopted in the two papers under review. The above-mentioned 1982 paper by the reviewer also showed that, if some open $\Omega \subset \mathbf{R}^n$ has the extension property, then the corresponding unitary representation $U(t)$ of $\mathbf{R}^n$ acts locally on $L^2(\Omega)$ by translation, i.e., $x \mapsto x + t$, when both x and $x + t$ are in Ω. When the point $x + t$ exits from Ω, it must return to Ω through a boundary operator. But the commutativity of the extension operators H_j suggests that the $\mathbf{R}^n$-translations will "wrap around" Ω in a way that makes Ω a tile for some set S of translations, and that S should be related to the spectrum Λ through the boundary operators. Unfortunately the latter are poorly understood, and the two papers under review are based instead on methods from harmonic analysis and finite abelian group combinatorics.

Both papers have a result to the effect that finite sets $A \subset \mathbf{R}^n$, which embed in the torus $\mathbf{R}^n/\mathbf{Z}^n$, will automatically have $\{e_\lambda\colon\ \lambda \in A + \mathbf{Z}^n\}$ as an orthogonal basis for $L^2(\Omega)$ if they are only assumed orthogonal, and it is further given that the cardinality of A, $\#A$, satisfies $\#A \geq \lambda_n(\Omega)$ (> 0), where $\lambda_n(\Omega)$ is the Lebesgue measure of Ω. The viewpoints and proofs in the two papers are very different, and they are both valuable and independent contributions.

Both papers provide a classification of the subsets $\Omega \subset \mathbf{R}^n$ of positive Lebesgue measure with spectrum Λ of the form $A + \mathbf{Z}^n$. The Pedersen classification is in terms of certain bundles over the torus $\mathbf{R}^n/\mathbf{Z}^n$, whereas the Lagarias-Wang classification is given in terms of an interesting new notion of universal spectrum. Their starting point is sets $\Omega \subset \mathbf{R}^n$ which tile with translation vectors $B + \mathbf{Z}^n$, where B is a subset of $(1/k_1)\mathbf{Z} \times (1/k_2)\mathbf{Z} \times \cdots \times (1/k_n)\mathbf{Z}$, i.e., $B \subset \mathbf{R}^n$ is rational with respect to the unit lattice $\mathbf{Z}^n$. But they also give a necessary and sufficient condition for such a rational periodic set to be a universal spectrum, and it is expressed in terms of set-factorizations $X \oplus Y = G$, where $G = \mathbf{Z}_{k_1} \times \mathbf{Z}_{k_2} \times \cdots \times \mathbf{Z}_{k_n}$ and $X = B \pmod{\mathbf{Z}^n}$.

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