Problems discussed by members of the NSF Focused Research Group on

Wavelets, Frames, and Operator Theory

in the Concentration Week, July 15–20, 2002, at Texas A&M University.

In addition to A. Aldroubi, C. Heil, D. Larson, P.E.T. Jorgensen, and G. Ólafsson from the FRG project, many invited participants from Texas, from around the country, from Europe, and from Singapore, took part in the workshop.

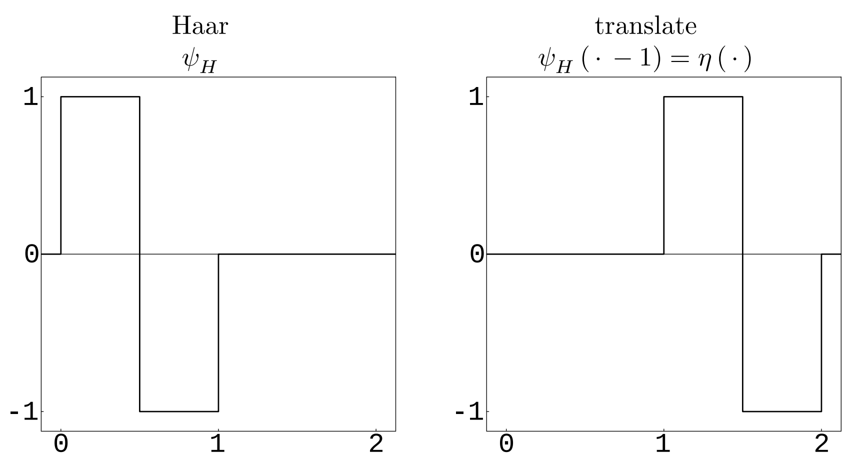
Manos Papadakis (7/19/02)

Let $\{V_j\}_j$ be a GFMRA of $L^2(\mathbb{R})$, and assume that M_0 is the low-pass filter associated with it. It is true that the $M_0(\xi)$'s define a bounded operator M for a.e. $\xi \in [0,1)$, and that $\operatorname{ess\,sup}\{\|M(\xi)\| : \xi \in [0,1)\} < +\infty$. Is it true that the function $\xi \to |M_0(\xi)|$ is a low-pass filter for a GFMRA? An answer may open the way to proving the connectivity of orthogonal wavelets, because all of them are associated with GFMRA's.

Dave Larson (7/20/02):

Let ψ_H be the Haar wavelet in $L^2(\mathbb{R})$. Let $\eta(t) = \psi_H(t-1)$, the 1-translate. Can ψ_H and η be connected by a continuous path in the L^2 metric so that all points in the path are real-valued orthonormal wavelets? Since the winding number jumps from ψ_H to η, we know from [BrJo02, Ch. 2] that such a homotopy path, real or complex, *cannot* be made with MRA-wavelets that have Lipschitz filters $m_i\colon \mathbb{T} \to \mathbb{C}$.

An illustration for D. Larson's question:



Hence

$$\int x\,|\psi_H(x)|^2\,dx = \frac{1}{2} \quad \text{and} \quad \int x\,|\psi_H(\,\cdot - 1)|^2\,dx = \frac{1}{2} + 1.$$

Connect ψ_H and $\eta = \psi_H(\,\cdot - 1)$ with a homotopy path in $L^2(\mathbb{R})$, real-valued. Describe the filters $\{m_t \mid 0 \le t \le 1\}$, $m_0 \sim \psi_H$, $m_1 \sim \psi_H(\,\cdot - 1) = \eta(\,\cdot\,)$.

A. Aldroubi—as recounted by P. Jorgensen

A thought: A law of large numbers for wavelets? Let some nice wavelet ψ be given. Then establish the following "limit laws":

(1) $\mathrm{Norm}_n(\underbrace{\psi * \cdots * \psi}_{n \text{ times}}) \xrightarrow[n\to\infty]{L^2(\mathbb{R}) \text{ (other?)}} \psi_{\text{Shannon}}$, where "Norm" means some renormalization, and

(2) Let ψ_{D_n} denote the Daubechies wavelets given by the Daubechies polynomials P_n [Dau92]. Show $\psi_{D_n} \xrightarrow[n\to\infty]{L^2(\mathbb{R})? \text{ (other?)}} \psi_{\text{Shannon}}$. Recall $\hat{\psi}_{\text{Shannon}} = \chi_{\left(-\pi,-\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right)}$.

Radu Balan 7/19/2002

I. *Issue:* Assume $A = (a_{ij})_{i,j\in\mathbb{Z}}$ is a bounded operator on $\ell^2(\mathbb{Z})$ and:

(i) It is invertible, say $B = (b_{ij})_{i,j} = A^{-1}$,

(ii) $\exists\, r \in \ell^1(\mathbb{Z})$ such that $|a_{ij}| \le r(i-j)$.

Question: Then: $\exists\, s \in \ell^1(\mathbb{Z})$ such that $|b_{ij}| \le s(i-j)$.

Comments: If true, it would be a noncommutative extension of Wiener's lemma.

What is known:

(1) Wiener lemma: The above holds true for Toeplitz A.

(2) Twisted Wiener lemma: OK for $a_{ij} = c_{i-j}e^{i(\varphi_i-\varphi_j)}$ (see [Gro01, Ch. 13], [FeGr97]).

(3) Other classes of matrices ...?

(4) If instead of $\ell^1(\mathbb{Z})$, one uses polynomial decay, the result is true and known as Jaffard's lemma.

(5) See also symmetric (noncommutative) normed algebras of operators ....

II. Find more uncertainty results for wavelet orthonormal bases.

Known: ([Bat97])

$$\{\psi_{mn}\} \text{ orthonormal basis}$$
$$\Downarrow$$
$$\int_{\mathbb{R}} (x-x_0)^2 |\psi(x)|^2\, dx \cdot \int_{\mathbb{R}} \xi^2 \left|\hat{\psi}(\xi)\right|^2 d\xi \ge \frac{3}{2}$$

with $\frac{3}{2}$ instead of $\frac{1}{2}$ (for example $x_0 = \int_{\mathbb{R}} x\,|\psi(x)|^2\, dx$).

([Bal98]) Similar if $\{\psi_{mn}\}$ Bessel sequence.

Chris Heil July 20, 2002

Q1. Are the continuous wavelets path-connected in the L^∞-norm?

(For the one-parameter family of examples, see the applet of Wim Sweldens [SwAp97], and C. Heil's papers [HeSt95], [Hei94], [CoHe92].)

Q2. Open problem due to R. Zalik: Does $\exists\, g \in L^2(\mathbb{R})$ and countably many points $\{a_n\}_{n=1}^\infty$ such that

$$(*) \qquad \{g(x-a_n)\}_{n=1}^\infty$$

is a Schauder basis for $L^2(\mathbb{R})$?

Remark: It is known that a system $(*)$ consisting only of translations cannot form a frame or Riesz basis for $L^2(\mathbb{R})$, but the question for Schauder basis seems to be much more delicate.

Q3. If $\left\{2^{\frac{n}{2}}\psi(2^n x - k)\right\}_{n,k\in\mathbb{Z}}$ is a wavelet frame that is not a Riesz basis, is it finitely linearly independent (i.e., is every finite subset still linearly independent)?

Q4. Are there "nice" necessary or sufficient conditions that ensure that a given scaling function is continuous?

Remark: For necessary *and* sufficient it is known that continuity is determined by the *joint spectral radius* of certain matrices, but this JSR is difficult to compute in general.

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Question 1: Let $\Lambda \subseteq \mathbb{R}^2$ be discrete, and suppose $\mathcal{D}^-(\Lambda) = 1$, where $\mathcal{D}^-(\Lambda)$ is the lower density of Λ When does there exist $\psi \in L^2(\mathbb{R})$ such that

$$\left\{x \mapsto e^{2\pi i b x}\psi(x-a)\right\}_{(a,b)\in\Lambda}$$

is a frame for $L^2(\mathbb{R})$? Or in other terms: Characterize those discrete sets $\Lambda \subseteq \mathbb{R}^2$ with $\mathcal{D}^-(\Lambda) = 1$ such that there exists $\psi \in L^2(\mathbb{R})$ so that

$$\left\{x \mapsto e^{2\pi i b x}\psi(x-a)\right\}_{(a,b)\in\Lambda}$$

is a frame for $L^2(\mathbb{R})$!

Question 2: Let $\Lambda \subset \mathbb{R}^+ \times \mathbb{R}$ be discrete and $\mathcal{D}^-_{\text{aff}}(\Lambda) > 0$. ($\mathcal{D}^-_{\text{aff}}(\Lambda)$ is the density with respect to the affine group.) Does there always exist $\psi \in L^2(\mathbb{R})$ such that

$$\left\{x \mapsto |a|^{-\frac{1}{2}}\,\psi\left(\frac{x}{a}-b\right)\right\}_{(a,b)\in\Lambda}$$

is a frame for $L^2(\mathbb{R})$?

Remarks of P. Jorgensen on connectivity of wavelets: The answer to the question of connectivity for wavelets depends on the context: MRA wavelets, GMRA wavelets, a connecting path in $L^2(\mathbb{R})$, a connecting path within some specified family of wavelet filters. There may be a path in $L^2(\mathbb{R})$ which connects two "nice" wavelets, but the path takes you outside the wavelet filters which satisfy some "mild" regularity condition, such as a Lipschitz property. A case in point is represented by ψ_H and its translate $\psi_H(\,\cdot - 1)$ where

$$\psi_H(x) = \begin{cases} 1, & 0 \le x < \frac{1}{2}, \\ -1, & \frac{1}{2} \le x < 1, \\ 0, & \text{all other } x \in \mathbb{R}. \end{cases}$$

For more details see [BrJo02], [Gar98], [Gar99], [BGRW99], [WUTAM].

For the "four-tap wavelet family" (W. Sweldens applet [SwAp97]) it is known that the continuous wavelets are connected [HeSt95], [Hei94], [CoHe92], and because of the joint spectral radius characterization of continuity, that the continuous ones form an open subset, but whether it is connected or not is completely open: the structure of the family of 4-tap examples is quite different from that of the 6-tap examples; see [BrJo02, Ch. 2]. *Terminology:* "4-tap" refers to masking coefficients a_0, a_1, a_2, a_3, and "6-tap" refers to a_0, a_1, a_2, a_3, a_4, a_5.

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