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★**Operator commutation relations.**

Commutation relations for operators, semigroups, and resolvents with applications to mathematical physics and representations of Lie groups.

Mathematics and its Applications.

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In the last thirty years the theory of continuous one-parameter groups and semigroups on Banach spaces has flourished. This theory, which has its historical roots in the dynamical description of quantum-mechanical systems, has subsequently found applications in many fields of mathematics. The main emphasis of this development has been on the analysis of single groups, or semigroups, and their generators. Less effort has been devoted to the study of the analytic structure of families of noncommuting groups and semigroups. Again such problems found their earliest motivation in quantum theory, notably with Heisenberg's formulation in terms of families of noncommuting observables. Problems of this nature form the principal object of the monograph under review and the authors attempt, with some success, to show that analysis of operator commutation relations leads to a unification of diverse areas of mathematics.

The 1959 paper of Nelson provided impetus to the study of the analytic structure of noncommutative systems such as Lie algebras acting on a Banach space. The central notion of this paper was the notion of analytic elements of the algebra. Subsequently great emphasis has been placed on analytic element methods and less attention has been devoted to the earlier less specific C^∞-methods. In this monograph an attempt is made to correct this imbalance by a systematic development of the C^∞-methods.

One of the main problems in commutation theory is that of exponentiation of a Lie algebra of unbounded operators on a Banach space. If $A_1, A_2, \cdots, A_n$ denotes a basis of the algebra then a necessary condition for exponentiation is that the closure of each A_i generates a strongly continuous one-parameter group. But a striking example of Nelson demonstrates that this condition is not sufficient. In this example $n = 2$ and A_1, A_2 are selfadjoint operators on a Hilbert space $L^2(M)$, which commute on a common dense domain D, but the unitary groups which they generate do not commute. One key aspect of this example is that D is not invariant under the unitary groups. Invariance properties of this kind appear indispensable for the fruitful interpretation of infinitesimal commutation relations, and such invariance properties play a key role in this monograph.

Another notion which is systematically analyzed is that of graph-density. One can associate with the basis $A_1, \cdots, A_n$ the C_1-Banach space $(E_1, \|\cdot\|_1)$ consisting of the elements E_1 of the Banach space E which are in the common domain of the A_i equipped with the norm $\|u\|_1 = \max\{\|u\|, \|A_i u\| \colon 1 \le i \le n\}$. Then one form of the graph-density condition, for a C_0-semigroup on E, is that the semigroup should restrict to a C_0-semigroup acting on $(E_1, \|\cdot\|_1)$. Alternative infinitesimal forms of graph-density are described and exploited in the analysis of commutation relations.

Conditions of approximate commutation also play an important part in the analysis. A typical notion of this kind for two operators A and B is that the ad-orbit of B under A, which is defined as the linear span of $\{(\mathrm{ad}\, A)^k(B) \colon 0 \le k < \infty\}$ where $(\mathrm{ad}\, A)(B) = AB - BA$, is finite-dimensional.

The combination of these ideas, domain invariance, graph-density, and approximate commutation, is shown to lead to a rich and interesting theory of commutation relations with a wide variety of applications. Some of the topics treated are mass splitting theorems in elementary particle physics, quantum-mechanical commutation relations, integration of Lie algebras, and unitary representations of noncompact groups. Despite this wealth of application the reader obtains the impression that there remains much to discover in commutation theory, and this monograph provides both motivation and a guide to the current state of knowledge.

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