Presentation for use with the textbook, Algorithm Design and Applications, by M. T. Goodrich and R. Tamassia, Wiley, 2015

Minimum Spanning Trees

Saguaros, Saguaro National Monument, 1941. Ansel Adams. U.S. government image. U.S. National Archives and Records Administration.

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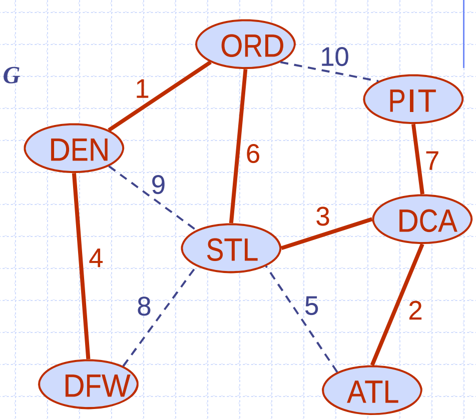
Minimum Spanning Trees

Spanning subgraph
- Subgraph of a graph ***G*** containing all the vertices of ***G***

Spanning tree
- Spanning subgraph that is itself a (free) tree

Minimum spanning tree (MST)
- Spanning tree of a weighted graph with minimum total edge weight

Applications
- Communications networks
- Transportation networks



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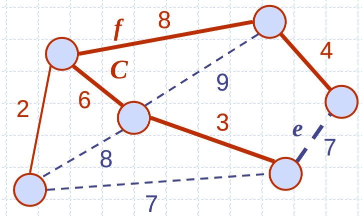
Cycle Property

Cycle Property:

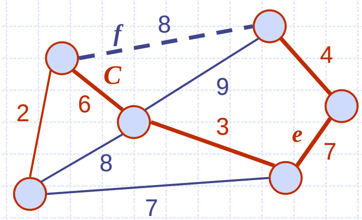
- Let T be a minimum spanning tree of a weighted graph G
- Let e be an edge of G that is not in T and C let be the cycle formed by e with T
- For every edge f of C, $weight(f) \leq weight(e)$

Proof:

- By contradiction
- If $weight(f) > weight(e)$ we can get a spanning tree of smaller weight by replacing e with f



Replacing f with e yields a better spanning tree



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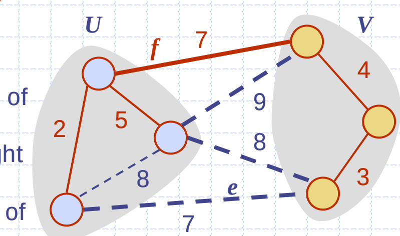
Partition Property

Partition Property:

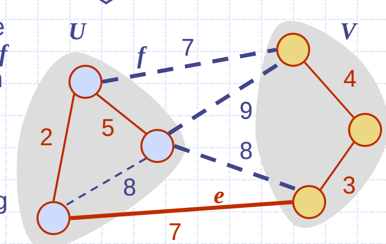
- Consider a partition of the vertices of G into subsets U and V
- Let e be an edge of minimum weight across the partition
- There is a minimum spanning tree of G containing edge e

Proof:

- Let T be an MST of G
- If T does not contain e, consider the cycle C formed by e with T and let f be an edge of C across the partition
- By the cycle property, $weight(f) \leq weight(e)$
- Thus, $weight(f) = weight(e)$
- We obtain another MST by replacing f with e



Replacing f with e yields another MST



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Prim-Jarnik's Algorithm

- Similar to Dijkstra's algorithm
- We pick an arbitrary vertex s and we grow the MST as a cloud of vertices, starting from s
- We store with each vertex v label $d(v)$ representing the smallest weight of an edge connecting v to a vertex in the cloud
- At each step:
  - We add to the cloud the vertex u outside the cloud with the smallest distance label
  - We update the labels of the vertices adjacent to u

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Prim-Jarnik's ~~Dijkstra's~~ Algorithm: Details

- **Input**: A weighted directed graph $G=(V, E)$, where $V=\{1, 2, \ldots, n\}$;
- **Output**: The distance from vertex 1 to every other vertex in G;

- 1. $X=\{1\}$; $Y \leftarrow V-\{1\}$; $D[1] \leftarrow 0$;
- 2. for $y \leftarrow 2$ to n
- 3. if (y is adjacent to 1) { $D[y] \leftarrow length[1, y]$; $p[y] \leftarrow 1$ }
- 4. else $D[y] \leftarrow \infty$;
- 5. for $j \leftarrow 2$ to n
- 6. Let $y \in Y$ be such that $D[y]$ is minimum;
- 7. $X \leftarrow X \cup \{y\}$; // add vertex y to X
- 8. $Y \leftarrow Y-\{y\}$; //delete vertex y from Y
- 9. for each edge (y, w)
- 10. if ($w \in Y$ and $\cancel{D[y]+}\ length[y, w] < D[w]$)
- 11. { $D[w] \leftarrow \cancel{D[y]+}\ length[y, w]$; $p[w] \leftarrow y$; }

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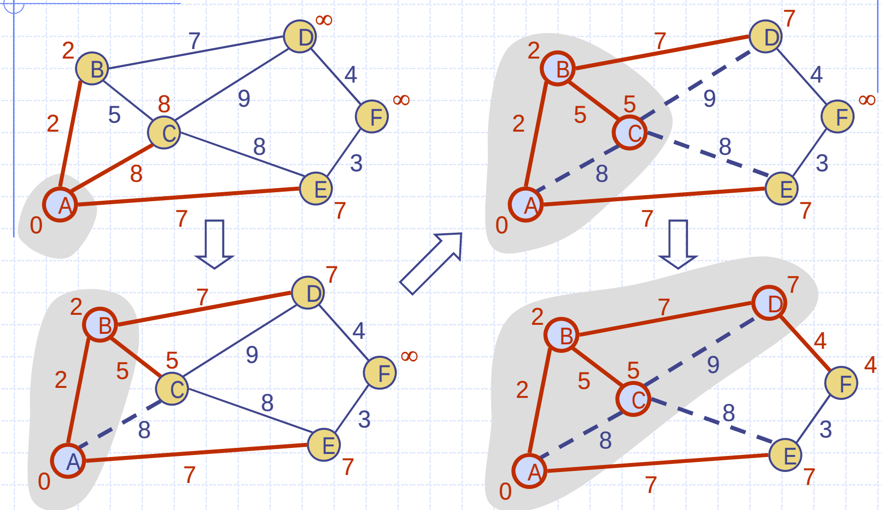
Example



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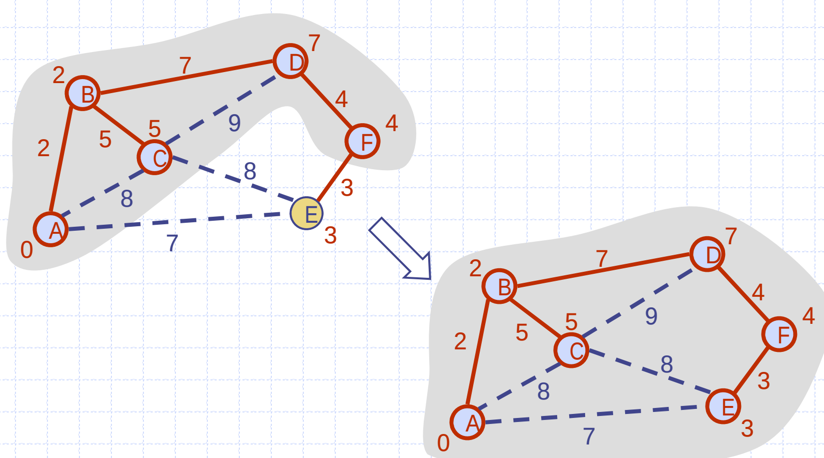
Example (contd.)



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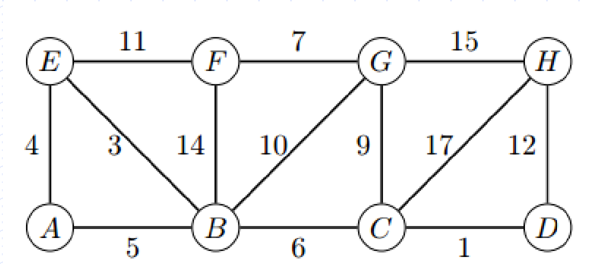
Possible Quiz Question

Find and draw the minimum spanning tree using Prim-Jarnik's Algorithm and list the nodes in the order of entering the cloud.



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Kruskal's Approach

- Maintain a partition of the vertices into clusters
  - Initially, single-vertex clusters
  - Keep an MST for each cluster
  - Merge "closest" clusters and their MSTs
- A priority queue stores the edges outside clusters
  - Key: weight
  - Element: edge
- At the end of the algorithm
  - One cluster and one MST

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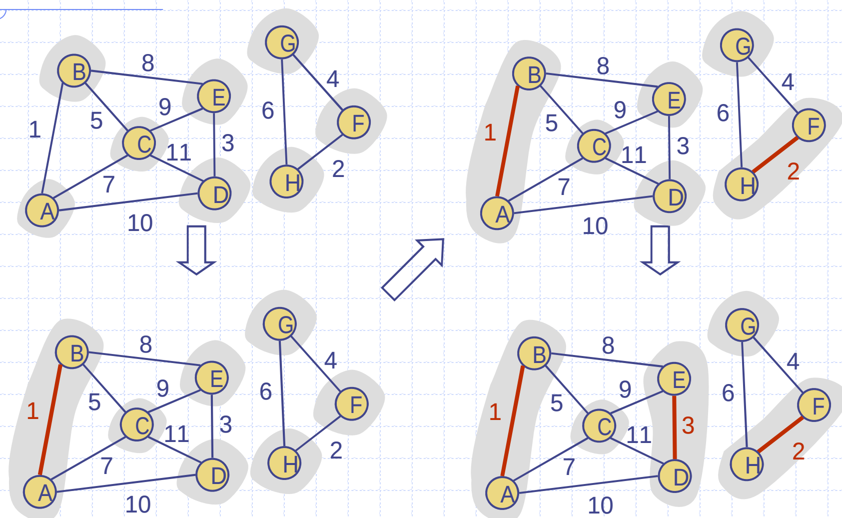
Example of Kruskal's Algorithm



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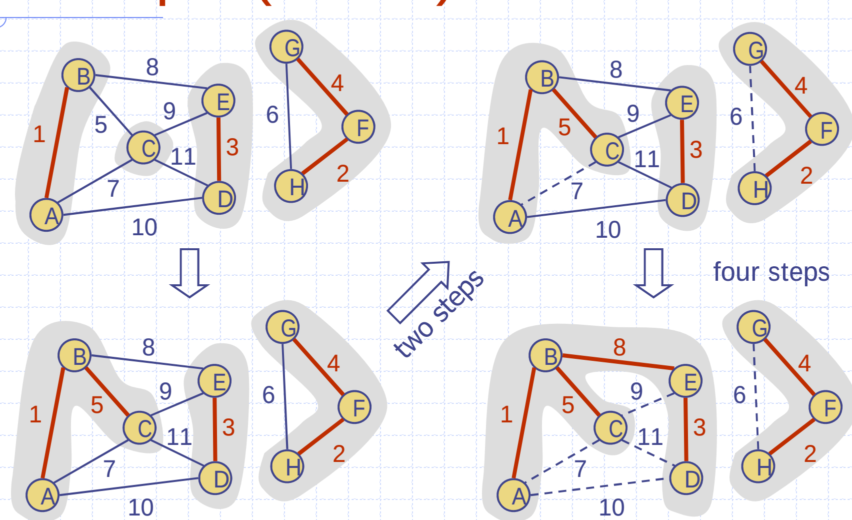
Example (contd.)



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Kruskal's Algorithm

Algorithm KruskalMST(G):

Input: A simple connected weighted graph G with n vertices and m edges
Output: A minimum spanning tree T for G

for each vertex v in G **do**
  Define an elementary cluster $C(v) \leftarrow \{v\}$.
Let Q be a priority queue storing the edges in G, using edge weights as keys
$T \leftarrow \emptyset$ // T will ultimately contain the edges of the MST
while T has fewer than $n-1$ edges **do**
  $(u, v) \leftarrow Q$.removeMin()
  Let $C(v)$ be the cluster containing v
  Let $C(u)$ be the cluster containing u
  if $C(v) \neq C(u)$ **then**
    Add edge (v, u) to T
    Merge $C(v)$ and $C(u)$ into one cluster, that is, union $C(v)$ and $C(u)$
return tree T

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Data Structure for Kruskal's Algorithm

- The algorithm maintains a forest of trees
- Sort all edges into non-decreasing order
- An edge is accepted if it connects distinct trees
- We need a data structure that maintains a partition, i.e., a collection of disjoint sets, with operations:
  - find(u): return the set storing u
  - union(A, B): replace sets A and B with their union

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Implementation with Union-Find

- Kruskal's Algorithm
  - Cluster merges as unions
  - Cluster locations as finds
- Running time $O(m \log n)$
  - Sorting: $O(m \log m) = O(m \log n)$
  - Union-Find operations: (practically) $O(n + m)$

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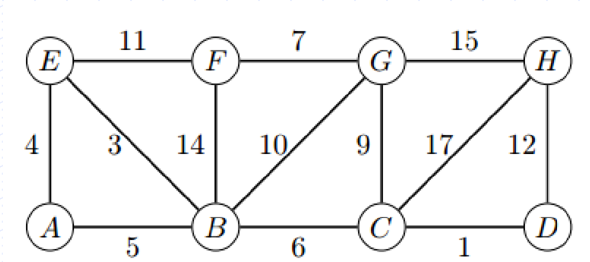
Possible Quiz Question

Find and draw the minimum spanning tree using Kruskal's Algorithm and list the edges in the MST in the order of entering the MST.



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Baruvka's Algorithm

- Like Kruskal's Algorithm, Baruvka's algorithm grows many clusters at once and maintains a forest T
- Each iteration of the while loop halves the number of connected components in forest T
- The running time: $O(m \log n)$

```
Algorithm BaruvkaMST(G)
  T ← V  {just the vertices of G}
  while T has fewer than n − 1 edges do
    for each connected component C in T do
      Let edge e be the smallest-weight edge from C to another component in T
      if e is not already in T then
        Add edge e to T
  return T
```

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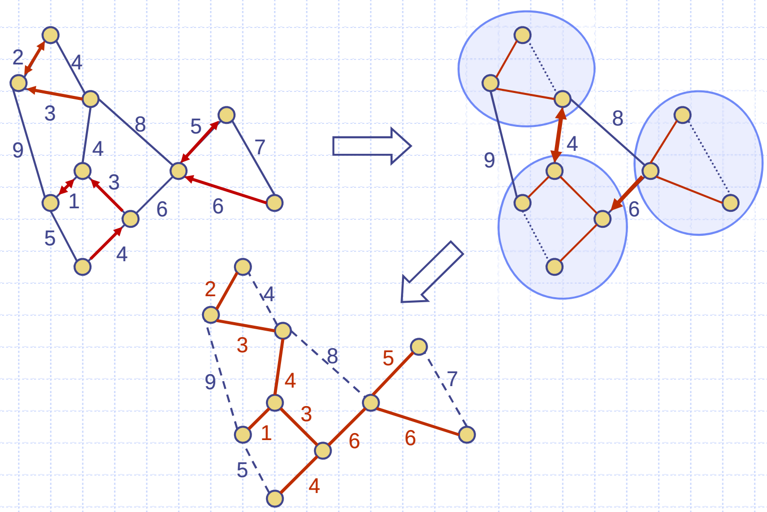
Example of Baruvka's Algorithm



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Data Structure for Baruvka's Algorithm

- Maintain the forest T subject to edge insertion, O(1) using linked list for T.
- Each vertex remembers its tree number, which is updated by DFS (O(n) time) after each round.
- For complexity analysis:
  - Minimum weight edges are obtained by going through all the edges in one tree (O(m) time) in each round.
  - Since the number of trees in T is halved after each round, there are at most O(log n) rounds. So the total complexity is O((n+m)log n).

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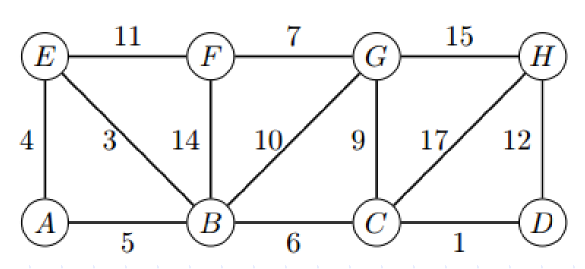
Possible Quiz Question

Find and draw the minimum spanning tree using Baruvka's Algorithm and list the edges in the MST in the order of entering the MST.



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