1. [40] Decide if the following languages are context-free. If yes, provide a context free grammar; if not, prove it by the pumping lemma and provide a decider for it.

(a) \( L_1 = \{a^i b^j a^i \mid i \geq j \geq 1 \} \);
This one is not CF. Using the standard pumping with \( z = a^N b^N a^N \) will do.
The description of the decider is skipped here.

(b) \( L_2 = \{a^i b^j a^i b^j \mid i \geq j \geq 1 \} \).
This one is not CF, either. Using the standard pumping with \( z = a^N b^N a^N b^N \) will do.
The description of the decider is skipped here.

2. [40] Decide with full arguments if the following problems are decidable (you may use any results given in the class):

(a) For any given standard Turing machine \( M \), any input word \( w \) and any integer number \( n \), will \( M \) runs on \( w \) for at least \( n \) steps?
This problem is decidable because the corresponding language
\[
L = \{\langle M, w, n \rangle \mid M \text{ runs on } w \text{ for at least } n \text{ steps}\},
\]
is total Turing-recognizable, where \( \langle M, w, n \rangle \) is the codes of \( M \), \( w \) and \( n \). That is, we can construct an algorithm to accept \( L \) as follows: The algorithm simulates \( M \) on \( w \) for at most \( n \) steps. If \( M \) doesn’t stop before \( n \) steps, the algorithm returns yes; otherwise no.

(b) For any given standard Turing machine \( M \), any input word \( w \) and any tape symbol \( x \), will \( x \) appear on the tape when \( M \) runs on \( w \)?
This problem is undecidable because the corresponding language
\[
L = \{\langle M, w, a \rangle \mid a \text{ appears on the tape when } M \text{ runs on } w\}
\]
is not total Turing-recognizable. Suppose \( L \) is total Turing-recognizable, then there exists an algorithm \( A \) to accept \( L \). Now we can construct an algorithm \( B \) to accept the universal language \( L_u = \{\langle M, w \rangle \mid w \in \text{L(M)}\} \) as follows: For the input \( \langle M, w \rangle \) to \( B \), \( B \) converts the input to the code of a new machine \( M' \) and then feeds \( \langle M', w, x \rangle \) to \( A \), where \( x \) is a new symbol not used by the encoding of \( \langle M, w \rangle \). \( B \) returns what \( A \) returns.
The property of \( M' \) is that \( M' \) prints \( x \) on the tape iff \( M \) accepts \( w \). That is, \( M' \) simulates \( M \) on \( w \). If \( M \) accepts \( w \), then \( M' \) prints \( x \) on the tape and then halts. Since \( A \) can tell if or not \( x \) will be printed on the tape, so \( B \) can tell if or not \( M \) accepts \( w \).

3. (40) Let \( \text{SET-SPLITTING} = \{\langle S, C \rangle \mid S \text{ is a finite set and } C = \{C_1, ..., C_k\} \text{ is a collection of subsets of } S, \text{ for some } k > 0, \text{ such that elements of } S \text{ can be colored red or blue so that no } C_i \text{ has all its elements colored with the same color.}\} \) Show that (a) \( \text{SET-SPLITTING} \) is in NP; (b) \( 3\text{SAT} \leq_p \text{SET-SPLITTING}. \)
This is one of the last homework problems.