Distributed Algorithms for Graph coloring
A few known results

• Any tree can be colored using two colors only

• Any graph whose maximum node degree is $\Delta$ can be colored using $(\Delta+1)$ colors

• Any planar graph can be colored using four colors, but no distributed algorithm is known and the centralized algorithm is also extremely cumbersome.

• Any tree of size $n$ can be colored using three colors in $\log^*(n)$ rounds (Cole and Vishkin)
O(log* n) coloring of a tree

Cole & Vishkin’s algorithm

Log*(n) is the smallest number of log-operations needed to bring n down to ≤2. For example, let n = one trillion. Now,

\[
\log (\text{one trillion}) = 40, \\
\log (\log (\text{one trillion})) = 5.322, \text{ and} \\
\log (\log (\log (\log (\text{one trillion})))) < 2.
\]

This means that log* (one trillion) = 4. This also illustrates that the function grows very slowly with the value of the argument.
Consider a rooted tree. The algorithm assumes that initially the color of each node is its id.

Each non-root node $v$ is aware of its parent $p(v)$. Interpret each color $c$ as a little-endian bit string $c_{k-1}c_{k-2}c_{k-3} \ldots c_0$, and let $|c|$ denote the size of the bit string.
\( O(\log^* n) \) coloring of a tree

**Program** `reduce` for a rooted tree

The root node labels itself with color 0 followed by bit 0 of its old color;

{Program for each non-root node \( v \) in a round}

**do** \( c(v) \geq 6 \rightarrow \)

{Let \( j \) = smallest index where the bit strings of \( c(v) \) and \( c(p(v)) \) differ}

new color \( c(v) := \) the bit string for \( j \) followed by bit \( j \) of \( c(v) \)

**od**
$O(\log^* n)$ coloring of a tree

(a) initial colors

(b) after one step
$O(\log^* n)$ coloring of a tree

The shift-down operation

The root picks a new color $< 6$ and different from its current color, and

Each non-root node $v$ concurrently executes $c(v) := c(p(v))$
$O(\log^* n)$ coloring of a tree
$O(\log^* n)$ coloring of a tree

Reduction of the color palette size from six to three

\[ z := 3; \]

\[ \text{do } z \leq 5 \rightarrow \]
\[ c(v) = z \rightarrow \text{pick a color from } \{0, 1, 2\} \text{ not used by the neighbors of } v; \]
\[ z := z + 1 \]

\[ \text{od} \]