Floating point Representation of Numbers

FP is useful for representing a number in a wide range: very small to very large. It is widely used in the scientific world. Consider, the following FP representation of a number

\[
\begin{array}{ll}
\text{Exponent E} & \text{significand F (also called mantissa)} \\
+/- & x x x x \\
& y y y y y y y y y y y y
\end{array}
\]

In \textit{decimal} it means (+/-) \(1. yyyyyyyyyyyyy \times 10^{xxx}\)

In \textit{binary}, it means (+/-) \(1. yyyyyyyyyyyyy \times 2^{xxx}\)

(The 1 is implied)
IEEE 754 single-precision (32 bits)

<table>
<thead>
<tr>
<th>s</th>
<th>xxxxxxxx</th>
<th>yyyyyyyyyyyyyyyyyyyyyyy</th>
<th>Single precision</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>8</td>
<td>23 bits</td>
<td></td>
</tr>
</tbody>
</table>

Largest = \(1.111\ldots \times 2^{+127} \approx 2 \times 10^{+38}\)

Smallest = \(1.000\ldots \times 2^{-128} \approx 1 \times 10^{-38}\)

These can be positive and negative, depending on \(s\).

(But there are exceptions too)

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IEEE 754 double precision (64 bits)

<table>
<thead>
<tr>
<th>S</th>
<th>exponent</th>
<th>significand</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>11 bits</td>
<td>52 bits</td>
</tr>
</tbody>
</table>

Largest = \(1.111\ldots \times 2^{+1023}\)

Smallest = \(1.000\ldots \times 2^{-1024}\)
Overflow and underflow in FP

An overflow occurs when the number is too large to fit in the frame. An underflow occurs when the number is too small to fit in the given frame.

How do we represent zero?

IEEE standards committee solved this by making zero a special case: if every bit is zero (the sign bit being irrelevant), then the number is considered zero.

Then how do we represent 1.0?
Then how do we represent 1.0?

It should have been $1.0 \times 2^0$ (same as 0)! The way out of this is that the interpretation of the exponent bits is not straightforward. The exponent of a single-precision float is "shift-127" encoded (biased representation), meaning that the actual exponent is (xxxxxxx minus 127). So thankfully, we can get an exponent of zero by storing 127.

- Exponent = 1111111 (i.e. 255) means 255-127 = 128
- Exponent = 0111111 (i.e. 127) means 127-127 = 0
- Exponent = 0000001 (i.e. 1) means 1-127 = -126
More on Biased Representation

The consequence of shift-127

Exponent = 00000000 (reserved for 0) can no more be used to represent the smallest number. We forego something at the lower end of the spectrum of representable exponents, (which could be $2^{-127}$). That said, it seems wise, to give up the smallest exponent instead of giving up the ability to represent 1 or zero!
More special cases

Zero is not the only "special case" float. There are also representations for positive and negative infinity, and for a not-a-number (NaN) value, for results that do not make sense (for example, non-real numbers, or the result of an operation like infinity times zero). How do these work? A number is infinite if every bit of the exponent is 1 (yes, we lose another one), and is NaN if every bit of the exponent is 1 plus any mantissa bits are 1. The sign bit still distinguishes +/-inf and +/-NaN. Here are a few sample floating point representations:

<table>
<thead>
<tr>
<th>Exponent</th>
<th>Mantissa</th>
<th>Object</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>Zero</td>
</tr>
<tr>
<td>0</td>
<td>Nonzero</td>
<td>Denormalized number*</td>
</tr>
<tr>
<td>1-254</td>
<td>Anything</td>
<td>+/- FP number</td>
</tr>
<tr>
<td>255</td>
<td>0</td>
<td>+/- infinity</td>
</tr>
<tr>
<td>255</td>
<td>Nonzero</td>
<td>NaN like 0/0 or ox inf</td>
</tr>
</tbody>
</table>

* Any non-zero number that is smaller than the smallest normal number is a denormalized number. The production of a denormal is sometimes called gradual underflow because it allows a calculation to lose precision slowly when the result is small.
Floating point operations in MIPS

32 separate single precision FP registers in MIPS

f0, f1, f2, … f31,

Can also be used as 16 double precision registers

f0, f2, f4, f30 (f0 means f0,f1 f2 means f2,f3)

These reside in a coprocessor C1 in the same package

Operations supported

add.s $f2, f4, f6 # f2 = f4 + f6 (single precision)
add.d $f2, f4, f6 # f2 = f4 + f6 (double precision)

(Also subtract, multiply, divide format are similar)

lwc1 $f1, 100($s2) # f1 = M [s2 + 100] (32-bit load)
mtc1 $t0, f0 # f0 = t0 (move to coprocessor 1)
mfc1 $t1, f1 # t1 = f1 (move from coprocessor 1)
Sample program

Evaluation of a Polynomial $a.x^2 + b.x + c$

```
# $f0 --- x 
# $f2 --- sum of terms

# Evaluate the quadratic
l.s $f2,a   # sum = a
mul.s $f2,$f2,$f0 # sum = ax

l.s $f4,b   # get b
add.s $f2,$f2,$f4 # sum = ax + b
mul.s $f2,$f2,$f0 # sum = (ax+b)x = ax^2 + bx

l.s $f4,c   # get c
add.s $f2,$f2,$f4 # sum = ax^2 + bx + c

.data
a: .float 1.0
b: .float 1.0
c: .float 1.0
```

Pseudo-instruction
**Floating Point Addition**

*Example using decimal*

\[ A = 9.999 \times 10^1, \quad B = 1.610 \times 10^{-1}, \quad A+B =? \]

**Step 1.** Align the smaller exponent with the larger one.

\[ B = 0.0161 \times 10^1 = 0.016 \times 10^1 \text{ (round off)} \]

**Step 2.** Add significands

\[ 9.999 + 0.016 = 10.015, \quad \text{so} \quad A+B = 10.015 \times 10^1 \]

**Step 3.** Normalize

\[ A+B = 1.0015 \times 10^2 \]

**Step 4.** Round off

\[ A+B = 1.002 \times 10^2 \]

Now, try to add 0.5 and -0.4375 in binary.
Floating Point Multiplication

Example using decimal

\[ A = 1.110 \times 10^{10}, \ B = 9.200 \times 10^{-5} \quad A \times B = ? \]

**Step 1.** Exponent of \( A \times B = 10 + (-5) = 5 \)

**Step 2.** Multiply significands

\[ 1.110 \times 9.200 = 10.212000 \]

**Step 3.** Normalize the product

\[ 10.212 \times 10^{5} = 1.0212 \times 10^{6} \]

**Step 4.** Round off

\[ A \times B = 1.021 \times 10^{6} \]

**Step 5.** Decide the sign of \( A \times B \) (+ x + = +)

So, \( A \times B = + 1.021 \times 10^{6} \)