Notes about Natural Numbers

Theorem: Let m,n∈**Nat** and m > 0. Then there exist unique natural numbers q (*quotient*) and r (*remainder*), 0 ≤ r < m, so that n = qm + r.

We may write n mod m = r, or rem(n,m) = r, and quotient(n,m) = q. The number d is a **divisor** of n if n mod d = 0, and n is a **multiple** of d (i.e., n = qd). If d is a divisor of both m and n, d is called a **common divisor** of m and n. If d is the largest common divisor of m and n, it is called the **greatest common divisor**, written as gcd(m,n).

Theorem: Let b>1 be a natural number (the base), Then for each n∈**Nat** with n > 0, there are natural numbers k ≥ 0 and $a_0, a_1, \dots, a_k$ with $0 \le a_i < b$ for $0 \le i \le k-1$ and $0 < a_k < b$ so that n is uniquely represented as $n = a_0 + a_1 b + a_2 b^2 + \dots + a_k b^k = \sum_{i=0}^{k} a_i b^i$.

In positional notation we write only the coefficients $a_0, a_1, \dots, a_k$, but in the reverse order (least significant coefficient to the right).

The preceding two theorems give rise to a straightforward algorithm for converting between bases.

Natural numbers (and Integers) are grouped into the following four categories based on their multiplication properties:

- zero — 0 alone (0 is a multiple of every integer)
- unit — u is a **unit** if xu = 1 for some integer x; 1 is the only unit for **Nat**, and {1, –1} are the units for **Int**
- prime — if p is not a unit and p = xy implies that either x or y is a unit, p is a **prime**
- composite — everything else (i.e., a product of two numbers that are neither a unit nor 0)

Prime Factorization Theorem: Any natural number n > 1 can be written uniquely as

$$n = p_1^{m_1} \, p_2^{m_2} \, \dots \, p_k^{m_k}$$

where k > 0, p_i is a prime and $m_i > 0$ ($1 \le i \le k$), and $1 < p_1 < p_2 < \dots < p_k$.

For real number x and natural number n, if n ≤ x < n+1, then floor(x) = n — **floor(x)** is the largest integer not exceeding x; if n < x ≤ n+1, then ceiling(x) = n+1 — **ceiling(x)** is the smallest integer not less than x.

The **factorial** of a natural number n, written n!, is defined to be n! = 1•2•3• … •(n–1)•n if n > 0, and 0! = 1. The factorial numbers are used to define the binomial numbers — for n,m∈**Nat** and n≥m, the **binomial number**, written $\binom{n}{m}$, is defined as $\binom{n}{m} = \frac{n!}{m!\,(n-m)!}$.

The number of permutations (or rearrangements) of n elements is n!, and the number of m element subsets of an n element set is $\binom{n}{m}$.

Binomial Expansion Theorem: For numbers x and y and n∈**Nat**,

$$(x+y)^n = \binom{n}{0} x^n y^0 + \binom{n}{1} x^{n-1} y^1 + \dots + \binom{n}{n-1} x^1 y^{n-1} + \binom{n}{n} x^0 y^n = \sum_{m=0}^{n} \binom{n}{m} x^{n-m} y^m.$$