Exam I — Sample Solutions

Problem 1.

Proof by induction:

Basis case: n=0

$\sum_{k=0}^{n}(2k+1) = \sum_{k=0}^{0}(2k+1) = 1 = 1^2 = (n+1)^2$ so true for n=0.

Induction step: assume true for n, prove for n+1

Inductive hypothesis: assume $\sum_{k=0}^{n}(2k+1) = (n+1)^2$.

Extend induction (prove for n+1):

$\sum_{k=0}^{n+1}(2k+1) = \sum_{k=0}^{n}(2k+1) + 2(n+1)+1$ by rearrangement

$= (n+1)^2+2n+3$ by inductive hypothesis

$= n^2+2n+1+2n+3 = n^2 +4n+4 = (n+2)^2 = ((n+1)+1)^2$ by algebra.

Therefore the induction is extended, and by induction the result is proven for all n.

Problem 2.

For parts (a) and (c) of this solution, assume the universe is E = {1, 2, 3}.

(a) (A ∩ B) ∪ (~A ∩ B) = A ∪ B is false —
for A = {1} and B = {2}, A ∪ B = {1, 2} while (A ∩ B) ∪ (~A ∩ B) = ∅ ∪ {2} = {2}.

(b) (A–B) – C = (A–C) – B is true —
If x∈(A–B) – C, then x∈(A–B) and x∉C and so x∈A and x∉B. Hence x∈(A–C) – B, and (A–B) – C ⊆ (A–C) – B. Conversely, if x∈(A–C) – B, then x∈(A–C) and x∉B and so x∈A and x∉C. Hence x∈(A–B) – C, and (A–C) – B ⊆ (A–B) – C and the proof is complete.

(c) (A–B) ∪ (B–A) = ~(A ∩ B) is false —
for A = {1} and B = {2}, ~(A ∩ B) = {1, 2, 3} while (A–B) ∪ (B–A) = {1, 2}.

Problem 3.

The relation R is an equivalence relation. Since m mod 2 = n mod 2, either both m and n must be even or both must be odd. This condition alone would yield the classes

[0] = {0, 2, 4, 6, … } and

[1] = {1, 3, 5, 7, … }.

But not all elements in one of these classes are equivalent under R. For instance, (0,2)∉R since (0 mod 3) ≠ (2 mod 3), and (0 mod 3) ≠ (4 mod 3), so (0,4)∉R. On the other hand, (0 mod 3) = (6 mod 3), so (0,6)∈R. The added requirement that (m mod 3) = (n mod 3) splits the even/odd classes above. Since there are three possible remainders from division by 3, each of these two classes splits into three and the six equivalence classes of R are:

[0] = {0, 6, 12, … } = {k•6 ∣ k∈N}	remainders 0,0 (from 2 and 3, respectively)
[2] = {2, 8, 14, … } = {2 + k•6 ∣ k∈N}	remainders 0,2 (from 2 and 3, respectively)
[4] = {4, 10, 16, … } = {4 + k•6 ∣ k∈N}	remainders 0,1 (from 2 and 3, respectively)

$[1] = \{1,7,13, \ldots\} = \{1 + k \cdot 6 \mid k \in N\}$	remainders 1,1 (from 2 and 3, respectively)
$[3] = \{3, 9, 15, \ldots\} = \{3 + k \cdot 6 \mid k \in N\}$	remainders 1,0 (from 2 and 3, respectively)
$[5] = \{5, 11, 17, \ldots\} = \{5 + k \cdot 6 \mid k \in N\}$	remainders 1,2 (from 2 and 3, respectively)

Problem 4.

(a) The function is defined as

$$f(n) = \begin{cases} n/2, & \text{for } n \text{ even} \\ \\ -((n+1)/2), & \text{for } n \text{ odd} \end{cases}$$

For an odd integer n, n+1 is even, and $-((n+1)/2)$ is the desired negative integer.

(b) f is 1-1 since if $f(n_1) = f(n_2)$, then n_1 and n_2 are either both even or both odd since otherwise $f(n_1)$ is negative and $f(n_2)$ is not, or vice-versa. Then in case both are even $n_1/2 = n_2/2$ implies $n_1 = n_2$. And in case both are odd, likewise $-((n_1+1)/2) = -((n_2+1)/2)$ implies $n_1 = n_2$.

Also f is onto Z since for any $n \geq 0$, $f(2n) = n$ since 2n is even, and for any $-n<0$, $2n-1$ is odd so $f(2n-1) = -(((2n-1)+1)/2) = -(2n/2) = -n$.