Notes on Graph Coloring

This topic assumes that we are discussing simple (non-oriented) graphs.

A graph G = (V,E) is **k-colorable** if there is a function c: V $\to$ {1, 2, … , k} (the *coloring function*) so that if (a,b)$\in$E, then c(a) ≠ c(b) — that is, adjacent nodes must have "different colors". The smallest number k so that G is k-colorable is called the **chromatic number** of G, written $\chi(G)$.

The **complete graph** on n nodes, K_n, has every possible edge. Its chromatic number is $\chi(K_n) = n$. For any tree T, $\chi(T) = 2$. Between these extremes, we find every possible variation. The only graphs G with $\chi(G) = 1$ are the graphs consisting entirely of isolated nodes — if there is even one edge, we must have $\chi(G) \geq 2$.

A graph is **planar** if it can be drawn in the plane with no edges crossing.

The **complete bipartite graph** $K_{m,n}$ has two subsets of nodes, one with m nodes and the other with n nodes. $K_{m,n}$ contains every possible edge from a node in one of the subsets to the nodes in the other, but no edges among nodes within the same subset. It therefore has m*n edges, while $\chi(K_{m,n}) = 2$ — color the nodes in one subset with one color, and the nodes in the other with the second.

Theorem: A graph G is planar if and only if it contains no subgraph "topologically equivalent" to $K_{3,3}$ or K_5.

Four Color Theorem: every planar graph is 4-colorable.