Semilocal Type Inference: Proof Appendix

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1 Soundness of $\vdash^D \text{wrt } \vdash^F$

- If $\Gamma \vdash^D t : T \rightsquigarrow e$ then $\Gamma \vdash^F e : T$
- If $\Gamma \vdash^I t : T \rightsquigarrow e \Rightarrow \sigma$ then $\Gamma, MV(\Gamma, \sigma(e)) \vdash^F \sigma(e) : \sigma(T)$
- If $\Gamma; \sigma \vdash^I (p, T) \cdot t' : T' \rightsquigarrow p' \Rightarrow \sigma'$
  and $\Gamma, MV(\Gamma, \sigma(p)) \vdash^F p : T$
  then $\Gamma, MV(\Gamma, \sigma'(p')) \vdash^F p' : T'$

1.1 Proof: Sound $\vdash^D$

Theorem 1. (Soundness of $\vdash^D \text{wrt } \vdash^F$): If $\Gamma \vdash^D t : T \rightsquigarrow e$ then $\Gamma \vdash^F e : T$

1.1.1 Case Var:

Directly from assumption

$\Gamma \vdash^F x : \Gamma(x) \text{ Var}$

1.1.2 Case AAbs:

Our assumed derivation is

$\begin{array}{c}
\Gamma, x : T \vdash^D t : T \rightsquigarrow e \\
\Gamma \vdash^D \lambda x : T.t : T \rightsquigarrow S \Rightarrow \lambda x : T. e
\end{array}$

AAbs

Invoking the IH on the premise we get $\Gamma, x : T \vdash^F e : S$ so we can conclude with

$\begin{array}{c}
\Gamma, x : T \vdash^F e : S \\
\Gamma \vdash^F \lambda x : T. e : T \rightsquigarrow S
\end{array}$

FAbs

1.1.3 Case Abs:

Similar to AAbs, invoking the IH on the premise (specialized to $\vdash^I$) and using FAbs.

1.1.4 Case TAbs

Similar to AAbs, invoking the IH on the premise and using FTAbs

1.1.5 Case TApp

Similar to AAbs and TAbs, invoking the IH on the premise (specialized to $\vdash^I$) and using FTApp.

1.1.6 Case AppSyn

Our assumed derivation is

$\begin{array}{c}
\Gamma \vdash^I t \cdot t' : T' \rightsquigarrow (e', \sigma_\emptyset) \\
MV(\Gamma, e') = \emptyset
\end{array}$

AppSyn

By mutual induction on the soundness of $\vdash^I$ on the first premise, we have $\Gamma, MV(\Gamma, e') \vdash^F \sigma_\emptyset(e') : \sigma_\emptyset(T')$, and after using the second condition and removing the identity substitution yields $\Gamma \vdash^F e' : T'$
1.1.7 Case $\text{AppChk}$

Our assumed derivation is

$$\Gamma \vdash^I t \; t' : T \rightsquigarrow (e', \sigma) \quad \text{MV}(\Gamma, p) = \text{MV}(\Gamma, T) = \text{dom}(\sigma) \quad \sigma(T') = T'' \quad \text{AppChk}$$

By mutual induction on the soundness of $\vdash^I$ we get

$$\Gamma, \text{MV}(\Gamma, \sigma'(e')) \vdash^F \sigma'(e') : \sigma(T')$$

From the second and fourth premise of our assumed derivation we know this is equivalent to

$$\Gamma \vdash^F \sigma(e') : T'$$

Which is what we need to conclude $\square$

1.2 Proof: Sound $\vdash^I$

Theorem 2. (Soundness of $\vdash^I$ wrt $\vdash^F$): If $\Gamma \vdash^I t : T \rightsquigarrow (p, \sigma)$ then $\Gamma, \text{MV}(\Gamma, \sigma(p)) \vdash^F \sigma(p) : \sigma(T)$

Pf. by induction on the assumed derivation.

1.2.1 Case $I\text{Head}$

Our assumed derivation is

$$t \neq t'[S] \land t \neq t_1 t_2 \quad \Gamma \vdash^I t : T \rightsquigarrow e \quad \text{IH}\text{ead}$$

By mutual induction on the soundness of $\vdash^I$ on the second premise we get

$$\Gamma \vdash^F e : T$$

Since $e$ is well-typed under $\Gamma$ it has no metavariables. Therefore, $\text{MV}(\Gamma, e) = \emptyset$, and we conclude

$$\Gamma, \emptyset \vdash^F \sigma(\emptyset)(e) : \sigma(\emptyset)(T)$$

1.2.2 Case $I\text{TApp}$:

Our assumed derivation is

$$\Gamma \vdash^I t : \forall X. T \rightsquigarrow (p, \sigma) \quad \text{ITApp}$$

By the IH on our premise we get

$$\Gamma, \text{MV}(\Gamma, \sigma(p)) \vdash^F \sigma(p) : \sigma(\forall X. T)$$

Implicit here is that $\Gamma \vdash S \text{ wf}$, so $\text{MV}(\Gamma, p[S]) = \text{MV}(\Gamma, p)$. Also, bound $X$ is fresh w.r.t. $\Gamma$, $p$, and $\sigma$ and so $\sigma(\forall X. T) = \forall X. \sigma(T)$. We conclude

$$\Gamma, \text{MV}(\Gamma, \sigma(p[S])) \vdash^F \sigma(p) : \forall X. \sigma(T) \quad \text{FTApp}$$
1.2.3 Case IApp:

Our assumed derivation is

\[ \frac{\Gamma \vdash t : T \leadsto (p, \sigma) \quad \Gamma \vdash (p : T, \sigma) \cdot t' : T' \leadsto (p', \sigma')}{\Gamma \vdash t \cdot t' : T' \leadsto (p', \sigma')} \]

IApp

By the IH on the premise we have

\[ \Gamma, MV(\Gamma, \sigma(p)) \vdash^F \sigma(p) : \sigma(T) \]

With this and with the second premise of our assumed derivation, we invoke mutual induction on the soundness of \( \vdash^I \) to get

\[ \Gamma, MV(\Gamma, \sigma'(p')) \vdash^F \sigma'(p') : \sigma'(T') \]

1.3 Soundness of \( \vdash^I \) wrt \( \vdash^F \)

**Theorem 3.** (Soundness of \( \vdash^I \) wrt \( \vdash^F \)) If \( \Gamma \vdash^I (p : T, \sigma) \cdot t' : T' \leadsto (p', \sigma') \) and \( \Gamma, MV(\Gamma, \sigma(p)) \vdash^F \sigma(p) : \sigma(T) \) then \( \Gamma, MV(\Gamma, \sigma'(p')) \vdash^F \sigma'(p') : \sigma'(T') \)

**Pf.** By induction on the assumed derivation of \( \vdash^I \)

1.3.1 Case \( \forall_I \)

Our assumed derivations are

\[ \sigma'' \in \{ \sigma, \sigma \circ [S/X] \} \quad \frac{\Gamma \vdash^I (p[X] : T, \sigma'') \cdot t' : T' \leadsto (p', \sigma')}{\Gamma \vdash^I (p : \forall X, T, \sigma) \cdot t' : T' \leadsto (p', \sigma')} \]

\( \forall_I \)

and

\[ \Gamma, MV(\Gamma, \sigma(p)) \vdash^F \sigma(p) : \sigma(\forall X. T) \]

We perform case analysis on \( \sigma'' \): either \( \sigma'' = \sigma \) or \( \sigma'' = \sigma \circ [S/X] \). If it is the former, then since \( X \) is fresh wrt \( \sigma \) we have \( MV(\Gamma, \sigma''(p[X])) = MV(\Gamma, \sigma(p)) \cup \{ X \} \) and \( \sigma''(p[X]) = \sigma(p[X]) \). We have by weakening

\[ \Gamma, MV(\Gamma, \sigma(p)) \vdash^F \sigma(p) : \sigma(\forall X. T) \]

\( \text{Weaken} \)

If \( \sigma'' = \sigma \circ [S/X] \) then we have \( MV(\Gamma, \sigma''(p[X])) = MV(\Gamma, \sigma(p)[S]) = MV(\Gamma, \sigma(p)) \) and we need only rewrite our second assumed derivation to \( \Gamma, MV(\Gamma, \sigma''(p[X])) \vdash^F \sigma''(p) : \sigma(\forall X. T) \)

In both cases, we can derive

\[ \frac{\Gamma, MV(\Gamma, \sigma''(p)) \vdash^F \sigma''(p) : \sigma(\forall X. T)}{\Gamma, MV(\Gamma, \sigma''(p[X])) \vdash^F \sigma''(p[X]) : \sigma''(T)} \]

TApp^F

We are now ready to invoke the IH with this and with the second premise of our assumed derivation of \( \vdash^I \) to derive

\[ \Gamma, MV(\Gamma, \sigma'(p')) \vdash^F p' : T' \]

which is what we need to conclude.
1.3.2 Case \( \cdot \top_I \)

Our assumed derivations are

\[
\begin{align*}
MV(\Gamma, \sigma(S)) &= \emptyset \\
\Gamma \vdash \Downarrow t' : \sigma(S) \rightsquigarrow e' \\
\Gamma \vdash \Downarrow (p : S \to T, \sigma) \cdot t' : T \rightsquigarrow (p e', \sigma) \cdot \Downarrow I
\end{align*}
\]

and

\[
\begin{align*}
\Gamma, MV(\Gamma, \sigma(p)) \vdash^F \sigma(p) : \sigma(S \to T)
\end{align*}
\]

By mutual induction on the soundness of \( \vdash \) \( \Downarrow \) \( \cdot \top_F \) and by weakening we have

\[
\begin{align*}
\Gamma \vdash \Downarrow t' : \sigma(S) \rightsquigarrow e' \\
\Gamma \vdash^F e' : \sigma(S) & \quad \text{Sound} \ \vdash \Downarrow \\
\Gamma, MV(\Gamma, \sigma(p)) \vdash^F \sigma(p e') : \sigma(S) & \quad \text{Weaken}
\end{align*}
\]

With this and our second assumption, the derivation of \( \vdash^F \), we can conclude

\[
\begin{align*}
\Gamma, MV(\Gamma, \sigma(p)) \vdash^F \sigma(p) : \sigma(S \to T) & \quad \Gamma, MV(\Gamma, \sigma(p)) \vdash^F e' : \sigma(S) \quad \text{App}
\end{align*}
\]

noting that since \( e' \) is well-typed under \( \Gamma \), \( \sigma(e') = e' \)

1.3.3 Case \( \cdot \top_I \)

Our assumed derivations are

\[
\begin{align*}
MV(\Gamma, \sigma(S)) &= Y \neq \emptyset \\
\Gamma \vdash \Downarrow t' : S' \rightsquigarrow e' \\
\Gamma \vdash \Downarrow (p : S \to T, \sigma) \cdot t' : [U/Y]T \rightsquigarrow ([U/Y]p e', \sigma) \cdot \top_I
\end{align*}
\]

and

\[
\begin{align*}
\Gamma, MV(\Gamma, \sigma(p)) \vdash^F \sigma(p) : \sigma(S \to T)
\end{align*}
\]

By mutual induction on the soundness of \( \vdash \) \( \Downarrow \) \( \cdot \top_F \) and weakening on the second premise of our assumed derivation of \( \vdash^F \), and after rewriting by our third such premise, we have

\[
\begin{align*}
\Gamma \vdash \Downarrow t' : [U/Y] \circ \sigma(S) \rightsquigarrow e' \\
\Gamma \vdash^F e' : [U/Y] \circ \sigma(S) & \quad \text{Sound} \ \vdash \Downarrow \\
\Gamma, MV(\Gamma, [U/Y] \circ \sigma(p)) \vdash^F [U/Y] \circ \sigma(p) : [U/Y] \circ \sigma(S) & \quad \text{Weaken}
\end{align*}
\]

Let \( \sigma'' = [U/Y] \circ \sigma \). By appeal to Lemma 5.1 on the typeability of substituting solutions in for metavariables we have

\[
\begin{align*}
\Gamma, MV(\Gamma, \sigma(p)) \vdash^F \sigma(p) : \sigma(S \to T) \\
\Gamma, MV(\Gamma, \sigma''(p)) \vdash^F \sigma''(p) : \sigma''(S \to T) & \quad \text{Lem. 5.1}
\end{align*}
\]

From this and rule App from \( \vdash^F \) we can derive

\[
\begin{align*}
\Gamma, MV(\Gamma, \sigma''(p)) \vdash^F \sigma''(p) : \sigma''(S \to T) & \quad \Gamma, MV(\Gamma, \sigma''(p)) \vdash^F e' : \sigma''(S) \quad \text{App}
\end{align*}
\]

\[
\begin{align*}
\Gamma, MV(\Gamma, \sigma([U/Y]p e')) \vdash^F \sigma([U/Y]p e') : \sigma([U/Y]T)
\end{align*}
\]
2 Soundness of $\vdash^A_wrt \vdash^D$

- If $\Gamma \vdash^A t : T \rightsquigarrow e$ then $\Gamma \vdash^\emptyset t : T \rightsquigarrow e$
- If $\Gamma \vdash^A t : T \rightsquigarrow e$ then $\Gamma \vdash^\emptyset t : T \rightsquigarrow e$
- If $\Gamma; P \vdash^A t : W \rightsquigarrow (p, \sigma)$ then $\Gamma \vdash^l t : [W] \rightsquigarrow (p, \sigma)$
- If $\Gamma; \sigma \vdash^A (p, W) \cdot t' : W' \rightsquigarrow (p', \sigma')$
  and $MV(\Gamma, p) \vdash := [W] := \Rightarrow P \Rightarrow (\sigma, W)$ with $\Gamma \vdash ? \Rightarrow P$ \text{\textit{wf}}
  then $\Gamma; \sigma \vdash^l (p, [W]) \cdot t' : [W'] \rightsquigarrow (p', \sigma')$

To show this, we need the following lemmas

- If $\Gamma; P \vdash^A t : W \rightsquigarrow (p, \sigma)$ then $MV(\Gamma, p) \vdash := [W] := \Rightarrow P \Rightarrow (\sigma, W)$
- If $\Xi \vdash := T := \Rightarrow (\sigma, W)$ then $\sigma(T) := \sigma([W])$
- If $\Xi \vdash := T := \Rightarrow (\sigma, W)$
  with $\Gamma \vdash P$ \text{\textit{wf}} and $\Gamma, \Xi \vdash T$ \text{\textit{wf}}
  then $\forall X \in \Xi, \sigma(X) = X$ or $\Gamma \vdash \sigma(S)$ \text{\textit{wf}}

2.1 Sound $\vdash^A\emptyset$

Thm. If $\Gamma \vdash^\emptyset t : T \rightsquigarrow e$ then $\Gamma \vdash^? t : T \rightsquigarrow e$

Pf. By induction on the assumed derivation $\Gamma \vdash^A\emptyset t : T \rightsquigarrow e$

2.1.1 Case $\text{Var}$:
Immediate from $\Gamma(x) = T$

2.1.2 Case $\text{AAbs}$:
We have as our assumption

\[ \Gamma, x : T \vdash^A\emptyset t : T \rightsquigarrow e \]

\[ \Gamma \vdash^\emptyset \lambda x : T. t : T \Rightarrow S \rightsquigarrow \lambda x : T. e \quad \text{AAbs} \]

Invoking the IH on the premise gives us

\[ \Gamma, x : T \vdash^A\emptyset t : T \rightsquigarrow e \]

\[ \Gamma, x : T \vdash^\emptyset e : T \quad \text{IH} \]

allowing us to conclude

\[ \Gamma \vdash^\emptyset \lambda x : T. t : T \Rightarrow S \rightsquigarrow \lambda x : T. e \quad \text{AAbs} \]

2.1.3 Case $\text{TAbs}$:
Similar to $\text{AAbs}$ above - invoke the IH on the premise and use (declarative) $\text{TAbs}$ to conclude.
2.1.4 Case $TApp$

Similar to $AAbs$ and $TAbs$ above.

2.1.5 Case $AppSyn$:

Our assumed derivation is

$$
\Gamma, ? \vdash t \; t' : T \rightsquigarrow e' \Rightarrow \sigma_{\emptyset} \\
MV(\Gamma, e') = \emptyset \quad \text{AppSyn}
$$

By mutual induction of sound $\vdash^A_\emptyset$ on the first premise, we have

$$
\Gamma \vdash^I t \; t' : [T] \rightsquigarrow (e', \sigma_{\emptyset})
$$

Since $[T] = T$ we can conclude

$$
\Gamma \vdash^I t \; t' : [T] \rightsquigarrow e' \Rightarrow \sigma_{\emptyset} \\
MV(\Gamma, e') = \emptyset \quad \text{AppSyn}
$$

\[\Box\]

2.2 Sound $\vdash^A_\emptyset$

**Thm.** If $\Gamma \vdash^A_\emptyset t : T \rightsquigarrow e$ then $\Gamma \vdash^\emptyset t : T \rightsquigarrow e$

**Pf.** By induction on the assumed derivation $\Gamma \vdash^A_\emptyset t : T \rightsquigarrow e$

2.2.1 Case $AAbs$:

Our assumed derivation is

$$
\Gamma, x : T \vdash^A_\emptyset t : T \rightsquigarrow e \\
\Gamma \vdash^A_\emptyset \lambda x : T. t : T \rightarrow S \rightsquigarrow \lambda x : T. e \quad \text{AAbs}
$$

By the IH on the premise and rule $AAbs$ from $\vdash^\emptyset$ we conclude

$$
\Gamma, x : T \vdash^A t : S \rightsquigarrow e \\
\Gamma \vdash^\emptyset \lambda x : T. t : T \rightarrow S \rightsquigarrow \lambda x : T. e \quad \text{IH}
$$

2.2.2 Case $Abs$:

Our assumed derivation is

$$
\Gamma, x : T \vdash^A_\emptyset t : S \rightsquigarrow e \\
\Gamma \vdash^A_\emptyset \lambda x : T. t : T \rightarrow S \rightsquigarrow \lambda x : T. e \quad \text{Abs}
$$

We invoke the IH on the premise to get

$$
\Gamma, x : T \vdash^A_\emptyset t : T \rightsquigarrow e \\
\Gamma \vdash^\emptyset \lambda x : T. t : T \rightarrow S \rightsquigarrow \lambda x : T. e \quad \text{Abs}
$$
2.2.3 Case \( TAbs \):

Our assumed derivation is

\[
\begin{align*}
\Gamma, X &\vdash^A t : T \rightsquigarrow e \\
\Gamma &\vdash^A \Lambda X. t : \forall X. T \rightsquigarrow \Lambda X. e
\end{align*}
\]

\( TAbs \)

By the IH and \( TAbs \) from \( \vdash^A \) we get

\[
\begin{align*}
\Gamma, X &\vdash^A t : T \rightsquigarrow e \\
\Gamma &\vdash^A \exists t : T \rightsquigarrow e
\end{align*}
\]

\( IH \)

\[
\begin{align*}
\Gamma &\vdash^A \Lambda X. t : \forall X. T \rightsquigarrow \Lambda X. e
\end{align*}
\]

\( TAbs \)

which is what we need to conclude.

2.2.4 Case \( ChkSyn \):

Our assumed derivation is

\[
\begin{align*}
t &\neq t' & \quad \Gamma &\vdash^A t : T \rightsquigarrow e \\
\Gamma &\vdash^A t : T \rightsquigarrow e
\end{align*}
\]

\( ChkSyn \)

By mutual induction on the soundness of \( \vdash^A \) we get

\[
\begin{align*}
t &\neq t' & \quad \Gamma &\vdash^A t : T \rightsquigarrow e \\
\Gamma &\vdash^A t : T \rightsquigarrow e
\end{align*}
\]

\( ChkSyn \)

2.2.5 Case \( AppChk \):

Our assumed derivation is

\[
\begin{align*}
\Gamma ; T &\vdash^A t t' : T' \rightsquigarrow (p', \sigma) \\
MV(\Gamma, p') & = MV(\Gamma, T') = dom(\sigma)
\end{align*}
\]

\( AppChk \)

By mutual induction of sound \( \vdash^A \) on the first premise, we have

\[
\begin{align*}
\Gamma &\vdash^A t t' : [T'] \rightsquigarrow (p', \sigma)
\end{align*}
\]

The last premise for \( AppChk \) we need is

\[
\sigma(T') = T
\]

To get this, we appeal to lemma \ref{lem:6.4} sound \( \vdash^A \) wrt \( \vdash^= \)

\[
\begin{align*}
MV(\Gamma, q) &\vdash^= T' := T \Rightarrow (\sigma, T')
\end{align*}
\]

The only rules that could form this match is \( := \), whose premise is

\[
\sigma(T') = T
\]

We can conclude
\[\Gamma \vdash t \ anno \ (p, \sigma) \quad MV(\Gamma, p) = MV(\Gamma, T') = \text{dom}(\sigma) \quad \sigma(T') = T\]

\[
\Gamma \vdash \sigma(p)
\]

\[\text{AppChk}\]

\section{2.3 Sound} \[\vdash_A^?\]

\textbf{Thm.} If \(\Gamma; P \vdash_A^? t : W \rightsquigarrow (p, \sigma)\) then \(\Gamma \vdash t : [W] \rightsquigarrow (p, \sigma)\)

\textbf{Pf.} By induction on the assumed derivation

\subsection*{2.3.1 Case ?Head:}

Our assumed derivation is

\[
t \neq t[S] \land t \neq t' \quad \Gamma \vdash_A^? t : T \rightsquigarrow e \quad \emptyset \vdash^? T := ? \Rightarrow P \Rightarrow (\sigma_{\emptyset}, W)
\]

\[
\Gamma; ? \vdash P \vdash_A^? t : W \rightsquigarrow e \Rightarrow \sigma_{\emptyset}
\]

\[?Head\]

By mutual induction on the soundness of \(\vdash_A^?\) we have

\[
\Gamma \vdash_A^? t : T \rightsquigarrow e \Rightarrow \vdash_A^? \quad \text{sound}
\]

We now to construct

\[
t \neq t[S] \land t \neq t' \quad \Gamma \vdash_A^? t : T \rightsquigarrow e
\]

\[
\Gamma \vdash^? t : T \rightsquigarrow e \Rightarrow \sigma_{\emptyset} \quad IHead
\]

\subsection*{2.3.2 Case ?TApp:}

Our assumed derivation is

\[
\Gamma; ? \vdash P \vdash_A^? t : \forall X = R.W \rightsquigarrow (p, \sigma) \quad R = X \lor R = S
\]

\[
\Gamma; ? \vdash P \vdash_A^? t[S] : [S/X]W \rightsquigarrow (p[S], \sigma)
\]

\[?TApp\]

By the IH on the first premise we have

\[
\Gamma \vdash^? t : \forall X. [W] \rightsquigarrow (p, \sigma)
\]

We can conclude with

\[
\Gamma \vdash^? t[S] : [S/X]W \rightsquigarrow (p[S], \sigma) \quad ITApx
\]

\subsection*{2.3.3 Case ?App}

Our assumed derivation is

\[
\Gamma; ? \vdash P \vdash_A^? t : W \rightsquigarrow (p, \sigma) \quad \Gamma; \sigma \vdash_A^? (p, W) \cdot t' : W' \rightsquigarrow (p', \sigma')
\]

\[
\Gamma; P \vdash_A^? t t' : W' \rightsquigarrow (p', \sigma') \quad ?App
\]

By the IH on the first premise we have

\[
\Gamma \vdash^? t : [W] \rightsquigarrow (p, \sigma)
\]
By mutual induction on the soundness of $\vdash: A$ on the second premise, we have

$$\Gamma; \sigma \vdash (p, [W]) \cdot t': [W'] \rightsquigarrow (p', \sigma)'$$

We can conclude with

$$\Gamma \vdash t : [W] \rightsquigarrow (p, \sigma) \quad \Gamma \vdash (p, [W]) \cdot t' : [W'] \rightsquigarrow (p', \sigma)'$$

$$\Gamma \vdash t \cdot t' : [W'] \rightsquigarrow (p', \sigma)' \quad IApp$$

$\square$

### 2.4 Sound $\vdash: A$ wrt $\vdash$

**Thm.** If $\Gamma; \sigma \vdash (p, W) \cdot t' : W' \rightsquigarrow (p', \sigma)'$

and $MV(\Gamma, p) \vdash [W] := ? \Rightarrow P \Rightarrow (\sigma, W)$ with $\Gamma \vdash ? \Rightarrow P \text{wf}$

then $\Gamma; \sigma \vdash (p, [W]) \cdot t' : [W'] \rightsquigarrow (p', \sigma)'$

**Pf.** By induction on the derivation of $\vdash: A$

#### 2.4.1 Case $\forall$

Our assumed derivation for $\vdash: A$ is

$$\Gamma; \sigma \circ [R/X] \vdash (p[R], W) \cdot t' : W' \rightsquigarrow (p', \sigma)'$$

and our assumed match is

$$MV(\Gamma, p) \vdash := \forall X. [W] := ? \Rightarrow P \Rightarrow (\sigma, \forall X = R.W)$$

The only rule that could result in this conclusion is $:= \forall$, whose premise is

$$MV(\Gamma, p), X \vdash := [W] := ? \Rightarrow P \Rightarrow (\sigma \circ [R/X], W)$$

We appeal to lemma 6.6 on the well-formedness of solutions in $\sigma \circ [R/X]$ to get $R = X$ or $\Gamma \vdash R \text{wf}$.

This makes $R$ a legal guess for our declarative system. Now we invoke the IH to get

$$\Gamma; \sigma \circ [R/X] \vdash (p[R], [W]) \cdot t' : [W'] \rightsquigarrow (p', \sigma)'$$

allowing us to conclude $\Gamma; \sigma \vdash (p, \forall X. [W]) \cdot t' : [W'] \rightsquigarrow (p', \sigma)'$

#### 2.4.2 Case $\cdot \downarrow$

Our assumed derivation is

$$MV(\Gamma, \sigma(S)) = \emptyset \quad \Gamma \vdash\downarrow t' : S \rightsquigarrow e$$

$$\Gamma; \sigma \vdash (p, S \Rightarrow W) \cdot t' : W \rightsquigarrow (p e', \sigma) \quad \downarrow$$

and our assumed match is

$$MV(\Gamma, p) \vdash := S \Rightarrow [W] := ? \Rightarrow P \Rightarrow (\sigma, S \Rightarrow W)$$

By mutual induction on the soundness of $\vdash: A$ on the second premise we have $\Gamma \vdash\downarrow t' : S \rightsquigarrow e$. We can conclude
\[\text{MV}(\Gamma, \sigma(S)) = \emptyset \quad \Gamma \vdash t : \sigma(S) \leadsto p' \quad \downarrow_M\]

\[\Gamma; \sigma \vdash \{p, S \Rightarrow [W]\} \cdot t' : [W] \leadsto (p \ e', \sigma) \quad \uparrow\]

2.4.3 Case \(\uparrow\)

Our assumed derivation is

\[\text{MV}(\Gamma, \sigma(S)) = \emptyset \quad \Gamma \vdash A \ t : [U/Y] \circ \sigma(S) \leadsto e\]

\[\Gamma; \sigma \vdash A \ p, S \Rightarrow [W] \cdot t' : [U/Y][W] \leadsto [U/Y]p \ e' \Rightarrow \sigma \quad \uparrow\]

and our assumed match is

\[\text{MV}(\Gamma, p) \vdash (S \Rightarrow [W] : ? \Rightarrow P \Rightarrow (\sigma, S \Rightarrow W))\]

By mutual induction on the soundness of \(\vdash A\), we have

\[\Gamma \vdash \emptyset \ t : [U/Y] \circ \sigma(S) \leadsto e\]

which allows us to conclude

\[\text{MV}(\Gamma, \sigma(S)) = \emptyset \quad \Gamma \vdash \emptyset \ t' : [U/Y] \circ \sigma(S) \leadsto e\]

\[\Gamma; \sigma \vdash \{p, S \Rightarrow [W]\} \cdot t' : [U/Y][W] \leadsto (U/Y)p \ e' \Rightarrow \sigma \quad \uparrow_M\]

3 Completeness of \(\vdash A\) wrt \(\vdash D\)

- If \(\Gamma \vdash \emptyset \ t : T \leadsto e\) then \(\Gamma \vdash A \emptyset \ t : T \leadsto e\)
- If \(\Gamma \vdash \emptyset \ t : T \leadsto e\) then \(\Gamma \vdash A \emptyset \ t : T \leadsto e\)
- If \(\Gamma \vdash \emptyset \ t : T^+ \leadsto p^+ \Rightarrow \sigma\)
  and \(\text{MV}(\Gamma, p^+) \vdash (S \Rightarrow T^+ : ? \Rightarrow P \Rightarrow (\sigma^A, W^+))\), where\(^1\)
  - \(\sigma \subseteq \sigma^A\)
  - \(P = ? \Rightarrow P\) when \(t \neq t'\)

then there exists \((p, T, \sigma^+)\) where

- \(\text{MV}(\Gamma, p) \vdash (S \Rightarrow T : ? \Rightarrow P \Rightarrow (\sigma^A, \sigma^+, W))\)
- \(\sigma^+(p, T, W) = (p^+, T^+, W^+)\) and \(\text{dom}(\sigma^+) = \text{MV}(\Gamma, p) - \text{MV}(\Gamma, p^+)\)
- \(\Gamma; \sigma \vdash A \ p, T \Rightarrow W \Rightarrow p \Rightarrow \sigma^A \circ \sigma^+\)

- If \(\Gamma; \sigma \vdash \emptyset \ (p^+, T^+) \cdot t' : T^+ \leadsto p^+ \Rightarrow \sigma'\) where
  - \(\text{MV}(\Gamma, p^+) \vdash (S \Rightarrow T^+ : ? \Rightarrow P \Rightarrow (\sigma^A, W^+))\) with \(\sigma' \subseteq \sigma^{A'}\)
  - and \(\text{MV}(\Gamma, p) \vdash (S \Rightarrow T : ? \Rightarrow P \Rightarrow (\sigma^A \circ \sigma^+, W))\) with \(\sigma \subseteq \sigma^A\)
  - and \(\sigma^+(p, T) = (p^+, T^+)\)

then exists \((p', T', W', \sigma'^+)\) where

- \(\text{MV}(\Gamma, p') \vdash (S \Rightarrow T', W' \leadsto p' \Rightarrow \sigma'^+)\) with \(\sigma' \subseteq \sigma^{A'}\)

\(^1\)The superscript + denotes only that the terms and types of the declarative system have some additional substitutions \(\sigma^+\) in them that the algorithmic rules would not have made.
\( \sigma'(p', T', W') = (p'^+, T'^+, W'^+) \), \( \text{dom}(\sigma'^+) = \text{MV}(\Gamma, p') - \text{MV}(\Gamma, p'^+) \)

and \( \Gamma; \sigma^A \circ \sigma'^+ \vdash_A (p', W) \cdot t' : W' \rightsquigarrow p' \Rightarrow \sigma'^A \circ \sigma'^+ \)

To show this, we need the following lemmas

- If \( X \vdash_? T := P := T \Rightarrow (\sigma, W) \) then \( \text{dom}(\sigma) \subseteq X \)
- If \( X \vdash_? [U/Y] T := P \Rightarrow (\sigma, W) \) and \( X \cap \text{FV}(U) = \emptyset \)
  then \( \overline{X}, \overline{Y} \vdash_? T := P \Rightarrow (\sigma \circ \sigma_{\overline{Y}}, W_{\overline{Y}}) \) where
  - \( \sigma_{\overline{Y}} \subseteq [U/Y] \)
  - \([U/Y]W_{\overline{Y}} = W \)
- If \( \Gamma; \sigma \vdash t : T \rightsquigarrow p'^+ \Rightarrow \sigma' \)
  and \( \text{MV}(\Gamma, p'^+) \vdash_? T'^+ := P \Rightarrow (\sigma'^A, W'^+) \), \( \sigma' \subseteq \sigma'^A \)
  then \( \text{MV}(\Gamma, p^+) \vdash_? T^+ := ? \Rightarrow (\sigma^A, W^+) \), \( \sigma \subseteq \sigma^A \)
- If \( \Gamma; \sigma \vdash (p, \forall X.S \rightarrow T Y) \cdot t' : T \rightsquigarrow p' \Rightarrow \sigma' \)
  then exists some \( \sigma_{\overline{Y}} \) with \( \overline{Y} \in \text{MV}(\Gamma, p) \cup \overline{X} \)
  such that \( \sigma_{\overline{Y}}(T'^{\overline{Y}}) = T'^{\overline{Y}} \)

3.1 Complete \( \vdash^A_0 \) wrt \( \vdash_? \)

Thm. If \( \Gamma \vdash_? t : T \rightsquigarrow e \) then \( \Gamma \vdash^A_0 t : T \rightsquigarrow e \)

Prf. By induction on the assumed derivation.

3.1.1 Case \textit{Var}: 

Our assumed derivation is 
\[
\Gamma \vdash_? x : \Gamma(x) \rightsquigarrow x \quad \text{Var}
\]

We get directly that 
\[
\Gamma \vdash^A_0 x : \Gamma(x) \rightsquigarrow x \quad \text{Var}
\]

3.1.2 Case \textit{Abs}: 

Our assumed derivation is 
\[
\Gamma, x : T \vdash_? t : S \rightsquigarrow e \quad \text{Abs}
\]

By the IH and rule \textit{Abs} of \( \vdash^A_0 \) we have 
\[
\Gamma, x : T \vdash_? t : S \rightsquigarrow e \quad \text{IH}
\]
\[
\Gamma \vdash^A_0 \lambda x : T. t : T \rightarrow S \rightsquigarrow \lambda x : T. e \quad \text{Abs}
\]
3.1.3 Case \(TAbs\):

Our assumed derivation is

\[
\frac{\Gamma, X \vdash t : T \leadsto e}{\Gamma \vdash \Lambda X. t : \forall X. T \leadsto \Lambda X. e} \quad TAbs
\]

From the IH and rule \(TAbs\) of \(\vdash^A\) we have

\[
\frac{\Gamma, X \vdash t : T \leadsto e}{\Gamma \vdash \forall X. T \leadsto \forall X. e} \quad IH
\]

\[
\frac{\Gamma \vdash \Lambda X. t : \forall X. T \leadsto \Lambda X. e}{\Gamma \vdash^A \Lambda X. t : \forall X. T \leadsto \Lambda X. e} \quad TAbs
\]

3.1.4 Case \(TApp\):

Our assumed derivation is

\[
\frac{\Gamma \vdash t : \forall X. T \leadsto e}{\Gamma \vdash t[S] : [S/X]T \leadsto e[S]} \quad TApp
\]

By the IH and by rule \(TApp\) of \(\vdash^A\) we have

\[
\frac{\Gamma \vdash t : \forall X. T \leadsto e}{\Gamma \vdash t[S] : [S/X]T \leadsto e[S]} \quad TApp
\]

3.1.5 Case \(AppSyn\)

Our assumed derivation is

\[
\frac{\Gamma \vdash t t' : T' \leadsto e' \Rightarrow \sigma \exists \quad MV(\Gamma, e') = \emptyset}{\Gamma \vdash t t' : T' \leadsto e'} \quad AppSyn
\]

To invoke mutual induction on the completeness of \(\vdash^A\) we must provide a match. This would be

\[
\emptyset \vdash := T' := \gamma \Rightarrow (\sigma \emptyset, T').
\]

It is immediate that the two preconditions hold. We now invoke complete \(\vdash^A\) to get \((p, T, W, \sigma^+)\) where

- \(MV(\Gamma, p) \vdash := T := \gamma \Rightarrow (\sigma \emptyset \circ \sigma^+, W)\)
  
  The only rule which forms conclusions like this is :=\(). From this we know the match is really.

- \(\sigma^+(p, T, W) = (e, T', T')\)
  
  This gives us \((p, T, W) = (e, T', T')\)

- \(\Gamma; ? \vdash^A t : W \leadsto p \Rightarrow \sigma \emptyset \circ \sigma^+\)
  
  which we rewrite to \(\Gamma; ? \vdash^A t : T' \leadsto e \Rightarrow \sigma \emptyset\)

We conclude \(\Gamma \vdash^A t t' : T' \leadsto e'\)

\(\square\)

3.2 Complete \(\vdash^A\) wrt \(\vdash\)

Thm. If \(\Gamma \vdash^A t : T \leadsto e\) then \(\Gamma \vdash t : T \leadsto e\)
3.2.1 Case \textit{AAbs}:

Our assumed derivation is
\[
\Gamma, x : T \vdash \_ : T \quad \vdash \_ e \\
\Gamma \vdash \lambda x : T. t : T \rightarrow S \rightsquigarrow \lambda x : T. e \quad \text{\textit{AAbs}}
\]

By the IH on the premise and by \textit{AAbs} from \(\vdash^A\) we have
\[
\Gamma, x : T \vdash \_ : S \quad e \\
\Gamma \vdash \lambda x : T. t : T \rightarrow S \rightsquigarrow \lambda x : T. e \quad \text{\textit{AAbs}}
\]

3.2.2 Case \textit{Abs}:

Our assumed derivation is
\[
\Gamma, x : T \vdash \_ : T \quad \vdash \_ e \\
\Gamma \vdash \lambda x : T. t : T \rightarrow S \rightsquigarrow \lambda x : T. e \quad \text{\textit{Abs}}
\]

By the IH on the premise and by \textit{Abs} from \(\vdash^A\) we have
\[
\Gamma, x : T \vdash \_ : S \quad e \\
\Gamma \vdash \lambda x : T. t : T \rightarrow S \rightsquigarrow \lambda x : T. e \quad \text{\textit{Abs}}
\]

3.2.3 Case \textit{TAbs}:

Our assumed derivation is
\[
\Gamma, X \vdash \_ : T \quad \vdash \_ e \\
\Gamma \vdash \lambda x : T. \forall X. T \rightsquigarrow \lambda X : T. e \quad \text{\textit{TAbs}}
\]

By the IH and rule \textit{TAbs} from \(\vdash^A\) we have
\[
\Gamma, X \vdash \_ : T \quad \vdash \_ e \\
\Gamma \vdash \lambda x : T. t : T \rightarrow S \rightsquigarrow \lambda x : T. e \quad \text{\textit{TAbs}}
\]

3.2.4 Case \textit{ChkSyn}:

Our assumed derivation is
\[
t \neq t' \quad t'' \quad \Gamma \vdash \_ : T \quad \vdash \_ e \\
\Gamma \vdash \_ : T \quad \vdash \_ e \quad \text{\textit{ChkSyn}}
\]

By mutual induction on the completeness of \(\vdash^A\) \textit{wrt} \(\vdash^A\) and rule \textit{ChkSyn} of \(\vdash^A\) we have
\[
t \neq t' \quad t'' \quad \Gamma \vdash \_ : T \quad \vdash \_ e \\
\Gamma \vdash \_ : T \quad \vdash \_ e \quad \text{\textit{Sound}} \\
\Gamma \vdash \_ : T \quad \vdash \_ e \quad \text{\textit{ChkSyn}}
\]
3.2.5 Case **AppChk**:

Our assumed derivation is:

\[ \begin{align*}
\Gamma \vdash^I t \ t' : T^+ \rightsquigarrow p^+ &\Rightarrow \sigma \\
MV(\Gamma, p^+) = MV(\Gamma, T^+) = dom(\sigma) &\Rightarrow \sigma(T^+) = T
\end{align*} \]

To invoke mutual induction on the completeness of \( \vdash^A \) we must provide a match. This would be

\[ \begin{align*}
\sigma(T^+) = T \\
\text{dom}(\sigma) = MV(\Gamma, T^+) &\Rightarrow FV(T) \cap MV(\Gamma, p^+) := T
\end{align*} \]

It is clear that the two preconditions on the completeness of \( \vdash^A \) hold, so we invoke it to get \((p', T', W', \sigma^+)\) where

- \( MV(\Gamma, p') \vdash^A := T' := T \Rightarrow (\sigma \circ \sigma^+, W') \)
- The only matching rule that could give us this conclusion is :=, which tells us \( W' = T' \)
- \( \sigma^+(p', T', W') = (p'^+, T'^+, T'^+) \) and \( \text{dom}(\sigma^+) = MV(\Gamma, p') - MV(\Gamma, p^+) \)
- This allows us to derive \( MV(\Gamma, p') = MV(\Gamma, T') = \text{dom}(\sigma \circ \sigma^+) \)
- \( \Gamma; T \vdash^A t : T' \rightsquigarrow p' \Rightarrow \sigma \circ \sigma^+ \)

We can therefore conclude

\[ \begin{align*}
\Gamma; T \vdash^A t \ t' : T' \rightsquigarrow p' &\Rightarrow \sigma \circ \sigma^+ \\
MV(\Gamma, p') = MV(\Gamma, T') &\Rightarrow \text{dom}(\sigma \circ \sigma^+) := T
\end{align*} \]

3.3 Complete \( \vdash^A \) wrt \( \vdash^I \)

If \( \Gamma \vdash^I t : T^+ \rightsquigarrow p^+ \Rightarrow \sigma \) and \( MV(\Gamma, p^+) \vdash^A := T^+ := P \Rightarrow (\sigma^A, W^+) \), where

- \( \sigma \subseteq \sigma^A \)
- \( P =? \Rightarrow P \) when \( t \neq t' \)

then there exists \((p, T, \sigma^+)\) where

- \( MV(\Gamma, p) \vdash^A := T := P \Rightarrow (\sigma^A \circ \sigma^+, W) \)
- \( \sigma^+(p, T, W) = (p^+, T^+, W^+), \text{dom}(\sigma^+) = MV(\Gamma, p) - MV(\Gamma, p^+) \)
- \( \Gamma; P \vdash^A t : W \rightsquigarrow p \Rightarrow \sigma^A \circ \sigma^+ \)

3.3.1 Case **IHead**

Our assumed derivation is

\[ \begin{align*}
t \neq t'[S] \land t \neq t' \Gamma \vdash^A t : T \rightsquigarrow e &\Rightarrow \sigma_{\emptyset} \\
\Gamma \vdash^I t : T \rightsquigarrow e &\Rightarrow \sigma_{\emptyset}
\end{align*} \]

Our assumed match is

\[ MV(\Gamma, e) \vdash^A := T := ? \Rightarrow P \Rightarrow (\sigma^A, W) \]

Since we know \( e \) is well-typed under \( \Gamma \), \( MV(\Gamma, e) = \emptyset \). Appealing to a lemma on the soundness of \( \text{dom}(\sigma^A) \) of matches, we get \( \text{dom}(\sigma^A) \subseteq \emptyset \), so \( \sigma^A = \sigma_{\emptyset} \). We rewrite our match to
\[ \emptyset \vdash T ::= ? \Rightarrow P \Rightarrow (\emptyset, W) \]

Now, invoke mutual induction on the completeness of \( \vdash^A \) to get

\[ \Gamma \vdash^A t : T \rightsquigarrow e \]

and choose \((e, T, W, \sigma_\emptyset)\) to meet the desired derivation and conditions

- \( MV(\Gamma, e) \vdash_:= T ::= ? \Rightarrow (\sigma_\emptyset, W) \)
- \( \sigma_\emptyset(e, T, W) = (e, T, W) \),
- \( \Gamma; ? \vdash P \vdash^A t : W \rightsquigarrow e \Rightarrow \sigma_\emptyset \)

### 3.3.2 ITApp

Our assumed derivation is

\[ \Gamma \vdash I t : \forall X.T^+ \rightsquigarrow p^+ \Rightarrow \sigma \]

\[ \Gamma \vdash I t[S] : [S/X]T^+ \rightsquigarrow p^+[S] \Rightarrow \sigma \]

and our assumed match is

\[ MV(\Gamma, p^+[S]) \vdash_:= [S/X]T^+ ::= ? \Rightarrow P \Rightarrow (\sigma^A, [S/X]W^+) \]

We appeal to a lemma on substitutions in the subject of matching to get

\[ MV(\Gamma, p^+[S]), X \vdash_:= T^+ ::= ? \Rightarrow P \Rightarrow (\sigma \circ [R/X], W_X^+) \]

With that match, we now can apply matching rule \( := \forall \) to get

\[ MV(\Gamma, p[S]) \vdash_:= \forall X. T^+ ::= ? \Rightarrow P \Rightarrow (\sigma, \forall X = R.W^+) \]

Lastly, we note that \( MV(\Gamma, p[S]) = MV(\Gamma, p) \), so we are able to invoke the IH to get \((p, T, W, \sigma^+)\) where

- \( MV(\Gamma, p) \vdash_:= T ::= ? \Rightarrow P \Rightarrow (\sigma^A \circ \sigma^+, W) \)
- \( \sigma^+(p, T, W) = (p^+, \forall X. T^+, \forall X = R.W^+) \)

\[ \text{and } \text{dom}(\sigma^+) = MV(\Gamma, p) - MV(\Gamma, p^+) \]

- \( \Gamma; ? \vdash P \vdash^A t : W \rightsquigarrow p \Rightarrow \sigma^A \circ \sigma^+ \)

Before we can finish the derivation of \( T.App \) we must deal with a subtle issue - what if \( T = Y \) and \( W = (Y, ? \Rightarrow P) \), with \( \sigma^+(Y) = \forall X = R.W^+ \)? This would prevent the algorithmic rules from applying a type application, and we’d be stuck!

Fortunately, we need only look at the match produced by the call to IH to sort this out. If \( T = Y \) then it could only be formed by \( := \text{Curr} \), yielding

\[ MV(\Gamma, p) \vdash_:= Y ::= ? \Rightarrow P \Rightarrow (\sigma^A \circ \sigma^+ = \sigma_\emptyset, (Y, ? \Rightarrow P)) \]

But now it’s impossible that \( \sigma^+(Y) = Y = \forall X = R.W^+ \). Therefore, we know that \( T \) has the form \( \forall X. T \) (we shadow the original \( T \) from here on out) and revisit our conclusions
• $MV(\Gamma, p) \vdash \forall X. T := ? \rightarrow P \Rightarrow (\sigma^A \circ \sigma^+, \forall X = R'.W)$

• $\sigma^+(p, \forall X. T, \forall X = R'.W) = (p^+, \forall X. T^+, \forall X = R.W^+)$
  and $\text{dom}(\sigma^+) = MV(\Gamma, p) - MV(\Gamma, p^+)$

  By a similar argument to the one above, we can see that $R' = X$ or $R' = S$

• $\Gamma; ? \vdash t : \forall X = R'.W \rightsquigarrow p \Rightarrow \sigma^A \circ \sigma^+$

This allows us to conclude

$\Gamma; ? \vdash t[S] : [S/X]W \rightsquigarrow p[S] \Rightarrow \sigma^A \circ \sigma^+$

3.3.3 Case IApp

Our assumed derivation is

$$
\text{IApp} \quad \frac{\Gamma \vdash t : T^+ \rightsquigarrow p^+ \Rightarrow \sigma \quad \Gamma; \sigma \vdash (p^+, T^+) \cdot t' : T'^+ \rightsquigarrow p'^+ \Rightarrow \sigma'}{\Gamma \vdash t \cdot t' : T'^+ \rightsquigarrow p'^+ \Rightarrow \sigma'}
$$

Our assumed match is

$MV(\Gamma, p^+) \vdash T^+ := P \Rightarrow (\sigma^A, W'^+)$

By lemma 7.1 on the generation of matches for the inputs of $\vdash^+$, we get from this and the second premise of the derivation

$MV(\Gamma, p^+) \vdash T^+ := ? \rightarrow P \Rightarrow (\sigma^A, W^+)$, with $\sigma \subseteq \sigma^A$.

This allows us to invoke the IH on the first premise of the derivation to get $(p, T, \sigma^+)$ where

• $MV(\Gamma, p) \vdash T := ? \rightarrow P \Rightarrow (\sigma^A \circ \sigma^+, W)$

• $\sigma^+(p, T, W) = (p^+, T^+, W^+)$ and $\text{dom}(\sigma^+) = MV(\Gamma, p) - MV(\Gamma, p^+)$

• $\Gamma; ? \vdash t : W \rightsquigarrow p \Rightarrow \sigma^A \circ \sigma^+$

The first two of these conditions, and the match we assumed, satisfy the preconditions of complete $\vdash^A$, allowing us to get $(p', T', W', \sigma'^+)$ where

• $MV(\Gamma, p') \vdash T' := P \Rightarrow (\sigma'^A \circ \sigma'^+, W')$ with $\sigma' \subseteq \sigma'^A$

• $\sigma'^+(p', T', W') = (p'^+, T'^+, W'^+)$, $\text{dom}(\sigma'^+) = MV(\Gamma, p') - MV(\Gamma, p'^+)$

• and $\Gamma; \sigma^A \circ \sigma^+ \vdash (p, W) \cdot t' : W' \rightsquigarrow p' \Rightarrow \sigma'^A \circ \sigma'^+$

This allows us to conclude

$\Gamma; P \vdash t \cdot t' : W' \rightsquigarrow p' \Rightarrow \sigma'^A \circ \sigma'^+$

3.4 Complete $\vdash^A$ wrt $\vdash^+$

Thm. If $\Gamma; \sigma \vdash (p^+, T^+) \cdot t' : T'^+ \rightsquigarrow p'^+ \Rightarrow \sigma'$ where

• $MV(\Gamma, p^+) \vdash T^+ := P \Rightarrow (\sigma^A, W'^+)$ with $\sigma \subseteq \sigma'^A$

• and $MV(\Gamma, p) \vdash T := ? \Rightarrow P \Rightarrow (\sigma^A \circ \sigma^+, W)$ with $\sigma \subseteq \sigma^A$

• and $\sigma^+(p, T) = (p^+, T^+)$, $\text{dom}(\sigma^+) = MV(\Gamma, p) - MV(\Gamma, p^+)$
then exists \((p', T', W', \sigma^+\)) where

- \(\text{MV}(\Gamma, p') \vdash_1 T' \leadsto P \Rightarrow (\sigma^A \circ \sigma^+, W')\) with \(\sigma' \subseteq \sigma^A\)
- \(\sigma^+(p', T', W') = (p^+, T^+, W^+), \text{dom}(\sigma^+) = \text{MV}(\Gamma, p') - \text{MV}(\Gamma, p^+)\)
- and \(\Gamma; \sigma^A \circ \sigma^+ \vdash : (p, W) \cdot t' : W' \leadsto p' \Rightarrow \sigma'^A \circ \sigma^+\)

### 3.4.1 Case \(\forall_I\)

Our assumed derivation is

\[
R \in \{X, U\}, \quad \Gamma \vdash U \text{ w.f.} \quad \Gamma; \sigma \circ [R/X] \vdash \land (p^+[R], T^+) \cdot t' : T^+ \leadsto p' \Rightarrow \sigma' \quad \forall_I
\]

Our assumed conditions are

- \(\text{MV}(\Gamma, p') \vdash_1 T' \leadsto P \Rightarrow (\sigma'^A, W'^+\) with \(\sigma' \subseteq \sigma'^A\)
- and \(\text{MV}(\Gamma, p) \vdash_1 T := ? \Rightarrow P \Rightarrow (\sigma^A \circ \sigma^+, W)\) with \(\sigma \subseteq \sigma^A\)
- and \(\sigma^+(p, T) = (p^+, \forall X.T^+), \text{dom}(\sigma^+) = \text{MV}(\Gamma, p') - \text{MV}(\Gamma, p^+)\)

As before, to make progress we need some way to reveal that \(T \neq Y\) for some \(Y \in \text{MV}(\Gamma, p)\). First, we note that it is easy to show that \(\forall X.T^+\) being in the application position of a judgment of \(\vdash\), it must really have the following form

\[
\forall X, \overline{X}.S^+ \rightarrow T^+_\overline{X}. \quad \text{(The reason for subscript \(\overline{Y}\) will become apparent a little later).}
\]

Now, if \(T = Y\) then the second match must have been formed by rule := \(\text{curr}\), which tells us

\[
\text{MV}(\Gamma, p) \vdash_1 Y := ? \Rightarrow P \Rightarrow (\sigma^A \circ \sigma^+, \sigma_\emptyset, (Y, ? \Rightarrow P))
\]

However, it is impossible that \(\sigma^+(Y) = \sigma_\emptyset(Y) = \forall X, \overline{X}.S^+ \rightarrow T^+_{\overline{X}}\). We can iterate this argument over each bound \(\overline{X}\) to get, finally, \(T\) looks like \(\forall X, \overline{X}.S \rightarrow T^+_{\overline{X}}\). Now, knowing this, we revisit the second and third assumed conditions on our derivation:

- \(\text{MV}(\Gamma, p) \vdash_1 \forall X, \overline{X}.S \rightarrow T^+_{\overline{X}} := ? \Rightarrow P \Rightarrow (\sigma^A \circ \sigma^+, \forall X = R', \overline{X} = R.S \rightarrow W_{\overline{X}})\)
  
  with \(\sigma \subseteq \sigma^A\)

- and \(\sigma^+(p, \forall X, \overline{X}.S \rightarrow T^+_{\overline{X}}) = (p^+, \forall X \overline{X}.S^+ \rightarrow T^+_{\overline{X}}), \text{dom}(\sigma^+) = \text{MV}(\Gamma, p') - \text{MV}(\Gamma, p^+)\)

We next need some way to relate guess \(R\) with decoration \(R'\), because in order to invoke the IH we need to satisfy precondition \(\sigma \circ [R/X] \subseteq \sigma^A \circ [R'/X]\). Given \(\sigma \subseteq \sigma^A\), it suffices to show \([R/X] \subseteq [R'/X]\). If \(R = X\) (i.e. the declarative rules decline to guess), then this is immediate. The setup above was all for the case where \(R = U\), since we need that the algorithm generated a decoration, and that it generated the \textit{same} decoration \(S\)!

We have by a small lemma that our \(\vdash\) derivation gives us some \(\sigma_{\overline{X}}\) with \(\overline{Y} \in \text{MV}(\Gamma, p) \sqcup \overline{X}\), such that \(\sigma_{\overline{X}}(T^+_{\overline{X}}) = T^+\)

If the declarative rules guessed \(S\), we have by an easy inductive argument, using the second premise of the derivation, that \([U/X] \in \sigma'\) (the set of guesses grows monotonically), so \([U/X] \in \sigma'^A\). Finally, we consider that since \(T^+ = \sigma_{\overline{X}} \circ \sigma(T')\), the match in our first condition is

\[
\text{MV}(\Gamma, p^+) \vdash_1 \sigma_{\overline{X}} \circ \sigma^+(T') := P \Rightarrow (\sigma'^A, W'^+)
\]
We invoke lemma 7.2 on un-substituting in the subject of matches to get

$$MV(\Gamma, p'), \exists T' := P \Rightarrow (\sigma'^A \circ \sigma'', W_T)$$

which we can repack into

$$MV(\Gamma, p) \vdash \forall X, X.S \rightarrow T' := P \Rightarrow (\sigma^A \circ \sigma^+, \forall X = U, X = R.S \rightarrow W_T)$$

Reuse of the same meta-variables in the result of the match is justified via lemma 7.3 on the uniqueness of matching solutions. To recap, we now meet the desired preconditions to invoke the IH

- $$MV(\Gamma, p') \vdash T'' := P \Rightarrow (\sigma'^A, W'')$$ with $$\sigma' \subseteq \sigma'^A$$
  This remains unmodified
- $$MV(\Gamma, p[R']) \vdash T := ? \Rightarrow P \Rightarrow (\sigma^A \circ [R'/X] \circ \sigma^+, W)$$ with $$\sigma^A \circ [R/X] \subseteq \sigma^A \circ [R'/X]$$
  Where we let $$T$$ again stand for $$X.S \rightarrow T'$$ and $$W$$ for $$X = R.S \rightarrow W_T$$
- $$\sigma^+ (p[R], T) = (p^+[R], T^+)$$

We invoke the IH to get $$(p', T', W', \sigma'^+)$$ where

- $$MV(\Gamma, p') \vdash T' := P \Rightarrow (\sigma'^A \circ \sigma'^+, W')$$ with $$\sigma' \subseteq \sigma'^A$$
- $$\sigma'^+(p', T', W') = (p'^+, T'^+, W'^+), \text{dom}(\sigma'^+) = MV(\Gamma, p') - MV(\Gamma, p'^+)$$
- $$\text{and } \Gamma; \sigma^A \circ [R'/X] \circ \sigma^+ \vdash_A (p[R'], W') \cdot t' : W' \Rightarrow p' \Rightarrow \sigma'^A \circ \sigma'^+$$

which is what we need to conclude with derivation

$$\Gamma; \sigma^A \circ \sigma^+ \vdash_A (p, \forall X = R'W) \cdot t' : W' \Rightarrow p' \Rightarrow \sigma'^A \circ \sigma'^+$$
3.4.2 Case \( \cdot \Downarrow_I \)

Our assumed derivation is

\[
MV(\Gamma, \sigma(S^+)) = \emptyset \quad \Gamma \vdash \cdot t' : \sigma(S^+) \rightsquigarrow e'.
\]

\[
\Gamma ; \sigma \vdash ! (p^+ , S^+ \Rightarrow T^+ ) \cdot t' : T^+ \rightsquigarrow p^+ e' \Rightarrow \sigma \cdot \Downarrow_I
\]

Our assumed conditions are

- \( MV(\Gamma, p e') \vdash := T^+ := P \Rightarrow (\sigma^{t^A}, W^{t^+}) \) with \( \sigma \subseteq \sigma^{t^A} \)
- and \( MV(\Gamma, p) \vdash := S \Rightarrow T :=? \Rightarrow P \Rightarrow (\sigma^A \circ \sigma^+, S \Rightarrow W) \) with \( \sigma \subseteq \sigma^A \)
  
  Again, reasoning that the subject of this match could not be some \( Y \in MV(\Gamma, p) \), it must be of the form \( S \Rightarrow T \)
- and \( \sigma^+(p, S \Rightarrow T) = (p^+, S^+ \rightarrow T^+) \)

We must now pick out a suitable \( (\sigma^{t^+}, p', T', W') \). Pick \( (\sigma^+, p e, T, W) \). Now we show the post-conditions of the theorem hold.

- \( MV(\Gamma, p e') \vdash := T := P \Rightarrow (\sigma^{t^A} \circ \sigma^+, W) \) with \( \sigma \subseteq \sigma^{t^A} \)
  
  From the second assumed condition, it is clear that
  
  \[
  MV(\Gamma, p e') \vdash := T := P \Rightarrow (\sigma^A \circ \sigma^+, W), \text{ so we need to show } \sigma^{t^A} = \sigma^A. \]
  
  By lemma \(6.3\) on the soundness of solutions from matching, we have
  
  \[
  MV(\Gamma, p e') \vdash := T^+ := P \Rightarrow (\sigma^A, \sigma^+(W)), \text{ and comparing to our first condition, uniqueness of solutions gives us } \sigma^A = \sigma^{t^A} \text{ and } \sigma^+(W) = W^{t^+}
  \]

- \( \sigma^+(p e', T, W) = (p^+ e', T^+, W^{t^+}), dom(\sigma^+) = MV(\Gamma, p e') - MV(\Gamma, p^+ e'), \) 
  
  Directly from assumptions and the equation in the point above, and from the fact that \( MV(\Gamma, e') = \emptyset \)
- and \( \Gamma ; \sigma^A \circ \sigma^+ \vdash ^A (p, S \Rightarrow W) \cdot t' : W \rightsquigarrow p e' \Rightarrow \sigma^{t^A} \circ \sigma^+ \)
  
  For this we invoke mutual induction on the completeness of \( \vdash ^A \) to get
  
  \[
  \Gamma \vdash ^A t' : \sigma^A \circ \sigma^+(S) \rightsquigarrow e', \text{ noting } \sigma^+(S) = S^+ \text{ and } MV(\Gamma, \sigma^A(S^+)) = \emptyset
  \]

Which allows us to conclude.
3.4.3 Case $\cdot \uparrow I$

Our assumed derivation is

$$MV(\Gamma, \sigma(S^+)) = \bar{Y}^+ \neq \emptyset \quad \Gamma \vdash_\emptyset t' : [U/Y]^+ \circ \sigma(S^+) \rightsquigarrow e'$$

$$\Gamma; \vdash_\emptyset (p^+, S^+ \Rightarrow T^+) \cdot t' : [U/Y]^+T^+ \rightsquigarrow [U/Y]^+p^+ e' \Rightarrow \sigma \cdot \uparrow I$$

Our assumed conditions are

- $MV(\Gamma, [U/Y]^+)p^+ e) \vdash_\emptyset [U/Y]^+p^+ e = P \Rightarrow (\sigma^{A'}, W'^+) \text{ with } \sigma \subseteq \sigma^{A'}$
- and $MV(\Gamma, p) \vdash_\emptyset S \Rightarrow T := T \Rightarrow P \Rightarrow (\sigma^A \circ \sigma^+, S \Rightarrow W) \text{ with } \sigma \subseteq \sigma^A$

Again, reasoning that the subject of this match must be $S \Rightarrow T$ and not some $Y \in MV(\Gamma, P)$

- and $\sigma^+(p, S \Rightarrow T) = (p^+, S^+ \Rightarrow T^+)$

We must pick a suitable $(p', T', W', \sigma^{A'})$. To do this, we must first ask what we know about $MV(\Gamma, \sigma^{A} \circ \sigma^+(S)) = \bar{Y}$ given $MV(\Gamma, \sigma(S^+)) = MV(\Gamma, \sigma \circ \sigma^+(S)) = \bar{Y}^+$ and $\sigma \subseteq \sigma^A$. Clearly $\bar{Y} \subseteq \bar{Y}^+$.

We consider the match from our first assumed condition. By lemma 7.2 on un-substitution on matching, we get

$$MV(\Gamma, p^+ e) \vdash_\emptyset T^+ := P \Rightarrow (\sigma^{A'} \circ \sigma_T, W'^+_{\bar{Y}}), \text{ where } \sigma_T(W'^+_{\bar{Y}}) = W'^+$$

We consider the match from our second assumed condition. By lemma 6.3 on re-substituting solutions and by inspecting its premise we get

$$MV(\Gamma, p^+ e) \vdash_\emptyset T^+ := P \Rightarrow (\sigma^A, \sigma^+(W))$$

And now, by uniqueness of solutions of matching (lemma 7.3), we get $(\sigma^A, \sigma^+(W)) = (\sigma^{A'} \circ \sigma_T, W'^+_{\bar{Y}})$. Let us call, for the sake of simplicity, the second component of both pairs $W^+$, and let $\sigma_T = [U/Y] - \sigma_T$

We return to the task of selecting $(p', T', W', \sigma^{A'})$. We pick

$$(\sigma_T(p) e', \sigma_T(T), \sigma_T(W), \sigma^+ \circ \sigma_T),$$

and now witness the following post-conditions:

- $MV(\Gamma, p') \vdash_\emptyset \sigma_T(T) := P \Rightarrow (\sigma^{A'} \circ \sigma^+ \circ \sigma_T, \sigma_T(W)) \text{ with } \sigma \subseteq \sigma^{A'}$
  
  This comes with some equational reasoning, noting that $\sigma^A = \sigma^{A'} \circ \sigma_T$ and that $dom(\sigma_T) \cap dom(\sigma^A \circ \sigma_T) = \emptyset$
  
- $\sigma^+(p', T', W') = (p'^+, T'^+, W'^+), \text{ dom}(\sigma^+) = MV(\Gamma, p') - MV(\Gamma, p^+)$
  
  Again with some equational reasoning. For example, $\sigma^+(T') = \sigma^+ \circ \sigma_T \circ \sigma_T = [U/Y]T^+ = T'^+$
  
- and $\Gamma; \vdash_\emptyset \sigma^A \circ \sigma^+ \vdash_\emptyset (p, W) \cdot t' : W' \Rightarrow p' \Rightarrow \sigma^{A'} \circ \sigma^+$
  
  This last piece requires some care. If $\bar{Y}^+ \neq \emptyset$, we know that the algorithm will try to derive

$$\Gamma \vdash_\emptyset t : \sigma_T(\sigma^A \circ \sigma^+(S)) \rightsquigarrow e'.$$

By an invocation of mutual induction on the completeness of $\vdash_\emptyset$ on the second premise, we know that the algorithm can derive

$$\Gamma \vdash_\emptyset t' : [U/Y]^+ \circ \sigma(S^+) \rightsquigarrow e'$$

which (by some equational reasoning) is what we need.
However, if $Y^+ = \emptyset$, the algorithm will actually try to check the term $t'$ against a fully known type. We need

$$\Gamma \vdash \sigma^A \circ \sigma^+ (S) \leadsto e'$$

By lemma 7.4 on the second premise of our assumed derivation we have

$$\Gamma \vdash (U/Y^+) \circ \sigma (S^+) \leadsto e'$$

By mutual induction on the completeness of $\vdash \sigma^A$ we get

$$\Gamma \vdash (U/Y^+) \circ \sigma (S^+) \leadsto e'$$

which, after a bit of equational reasoning on the substitutions, is what we need. So in either case, we are able to conclude

$$\Gamma; \sigma^A \circ \sigma^+ \vdash (p, W) \cdot t' : W' \leadsto p' \Rightarrow \sigma'^A \circ \sigma'^+$$

$\square$
4 Qualified Completeness of $\vdash^D$ wrt $\vdash^F$

Let $e_P$ be a fully-elaborated System F term and $t_P$ be a partial erasure of $e_P$. Further assume that the following conditions hold for all corresponding sub-expressions $e$ and $t$ in $e_P$ and $t_P$:

1. If $e = \lambda x : S.e'$ then $t = \lambda x : S.t'$
2. If $e = e' e''$, and $e$ occurs as a non-applicand in $e_P$
   and $\Gamma \vdash t : T \rightsquigarrow p \Rightarrow \sigma_\emptyset$
   then $MV(\Gamma, p) = \emptyset$
3. If $e = e' e''$, $t = t' t''$
   and $\Gamma \vdash t' : T' \rightsquigarrow p \Rightarrow \sigma_\emptyset$
   then $T = \forall X. S \rightarrow T$ for some $X, S, T$

Under these conditions, we can show the following:

- If $e$ occurs as a non-applicand in $e_P$
  or for all $e', e''$, and $S$ $e \neq e' e'' \wedge e \neq e'[S]$
  and $\Gamma \vdash^F e : T$ then $\Gamma \vdash_\emptyset t : T \rightsquigarrow e$

- If $e$ occurs as an applicand $e_P$ and $e = e' [S]$, $t = t' [S]$ with $e' \neq e''[S]$
  and $\Gamma \vdash^F e : T'$
  then $\Gamma \vdash^I t : T \rightsquigarrow p \Rightarrow \sigma_\emptyset$ with some $\sigma$ such that $dom(\sigma) = MV(\Gamma, p)$ and $\sigma(p, T) = (e, T')$

- If $e = e' e''$ and $\Gamma \vdash^F e : T'$ then $\Gamma \vdash^I t : T \rightsquigarrow p' \Rightarrow \sigma_\emptyset$
  with some $\sigma$ such that $dom(\sigma) = MV(\Gamma, p')$ and $\sigma(p', T) = (e, T')$

- If for all $e', e''$, and $S$, if $e \neq e' e''$ and $e \neq e'[S]$
  and $\Gamma \vdash^F e : T$ then $\Gamma \vdash^I t : T \rightsquigarrow e \Rightarrow \sigma_\emptyset$

- If
  - $\Gamma \vdash^F e \ [U_1] \ [U_2] : S' \rightarrow T'$ and $\Gamma \vdash_\emptyset t' : S' \rightsquigarrow e'$
  - and some $\sigma$ with $dom(\sigma) = MV(\Gamma, p \ [X_1] \ [X_2])$
    where $\sigma(p \ [X_1] \ [X_2], S \rightarrow T) = (e \ [U_1] \ [U_2], S' \rightarrow T')$
    (and $(|X_1|, |X_2|) = (|U_1|, |U_2|)$)
  then
  - $\Gamma ; \sigma_\emptyset \vdash^I (p \ [X_1], \forall [X_2]. S \rightarrow T) \cdot t' : T'' \rightsquigarrow p' \Rightarrow \sigma_\emptyset$
  - with some $\sigma'$ with $dom(\sigma') = MV(\Gamma, p')$ and where $\sigma(p', T'') = (e \ [U_1] \ [U_2] e', T')$

4.1 Complete $\vdash_\emptyset$ wrt $\vdash^F$

*Thm.* Under the given qualifications, if $e$ occurs as a non-applicand in $e_P$
  or for all $e', e''$, and $S$ $e \neq e' e'' \wedge e \neq e'[S]$
  and $\Gamma \vdash^F e : T$ then $\Gamma \vdash_\emptyset t : T \rightsquigarrow e$

*Pf.* By induction on the assumed derivation
4.1.1 Case \textit{Var}

Our assumed derivation is

\[
\Gamma \vdash F \ x : \Gamma (x) \quad \text{Var}
\]

There is only one partial erasure of \(x\). We apply rule \textit{Var} of \(\vdash \hat{\cdot}\) to conclude

\[
\Gamma \vdash \hat{\cdot} \ x : \Gamma (x) \rightarrow x \quad \text{Var}
\]

4.1.2 Case \textit{Abs}

Our assumed derivation is

\[
\Gamma, x : T \vdash F \ e : S \quad AAbs
\]

By our first assumed qualification, we have that our partial erasure \(t'\) of \(\lambda x : T.e\) has the form \(\lambda x : T.t\) for some partial erasure \(t\) of \(e\). We invoke the IH and conclude

\[
\Gamma, x : T \vdash \hat{\cdot} \ t : S \leadsto e \quad IH
\]

\[
\Gamma \vdash \hat{\cdot} \ \lambda x : T.t : T \rightarrow S \leadsto \lambda x : T.e \quad AAbs
\]

4.1.3 Case \textit{TAbs}

Our assumed derivation is

\[
\Gamma, X \vdash F \ e : T \quad TAbs
\]

We have a partial erasure \(\Lambda X.t\) of \(\Lambda X.e\), meaning that \(t\) is a partial erasure of \(e\). We invoke the IH to conclude

\[
\Gamma, X \vdash \hat{\cdot} \ t : T \leadsto e \quad IH
\]

\[
\Gamma \vdash \hat{\cdot} \ \Lambda X.t : \forall X.T \leadsto \Lambda X.e \quad TAbs
\]

4.1.4 Case \textit{TApp}

Our assumed derivation is

\[
\Gamma \vdash F \ e : \forall X.T \quad TApp
\]

By assumption, \(e[S]\) occurs in a non-applicand position in \(e_P\). This means that its erasure \(t\) corresponding to the same position in \(t_P\) has form \(t = t'[S]\) by the definition of erasure (we only erase type arguments between term to term applications).

Because \(e[S]\) is in an applicand position, neither is \(e\). Therefore, we can invoke the IH to conclude

\[
\Gamma \vdash \hat{\cdot} \ e : \forall X.T \quad IH
\]

\[
\Gamma \vdash \hat{\cdot} \ t : \forall X.T \quad TApp
\]

\[^2\text{Recall that by “applicand” we specifically mean regarding term to term applications}\]
4.1.5 Case App

Our assumed derivation is

\[
\begin{array}{c}
\Gamma \vdash F e : S' \to T' \\
\Gamma \vdash F e' : S'
\end{array} \quad \text{App}
\]

Since the elaborated expression in question is \(e e'\) we know that its erasure must be of the form \(t t'\). We invoke mutual induction for completeness of \(\vdash I\) for applications to get

\[
\Gamma \vdash I t t' : T \leadsto p' \Rightarrow \sigma_\emptyset
\]

with \(\sigma\) such that

- \(\text{dom}(\sigma) = MV(\Gamma, p)\)
- \(\sigma(p', T) = (e e', T')\)

Now, by assumption \(e e'\) occurs in a non-applicand position in \(e_P\). By qualification #2 we have \(MV(\Gamma, p) = \emptyset\). We use this to rewrite the post-conditions of our invocation of mutual induction above:

- \(\text{dom}(\sigma) = MV(\Gamma, p') = \emptyset \Rightarrow \sigma = \sigma_\emptyset\)
- \(\sigma(p', T) = \sigma_\emptyset(p', T) = (p', T) = (e e', T')\)

We can now conclude

\[
\Gamma \vdash I t t' : T' \leadsto e e' \Rightarrow \sigma_\emptyset \\
\Gamma \vdash \emptyset t t' : T' \leadsto e e' \Rightarrow \sigma_\emptyset \\
\Gamma \vdash \emptyset t t' : T' \leadsto e e' \Rightarrow \sigma_\emptyset
\]

\[\text{AppSyn}\]

\[\Box\]

4.2 Complete \(\vdash I\) wrt \(\vdash F\) (Application Head)

Thm. If for all \(e', e'', S\), if \(e \neq e e'\) and \(e \neq e'\) \[S\],
and \(\Gamma \vdash F e : T\) then \(\Gamma \vdash I t : T \leadsto e \Rightarrow \sigma_\emptyset\)

Pf. By case analysis on \(e\)

4.2.1 Case \(e = x\)

Our assumed derivation is

\[\Gamma \vdash x : \Gamma(x)\]

We appeal to mutual induction on the completeness of \(\vdash \emptyset\) wrt \(\vdash F\) to get

\[\Gamma \vdash \emptyset x : \Gamma(x) \leadsto x\]

and from this derive

\[
\begin{array}{c}
x \neq t[S] \land x \neq t t'
\end{array} \quad \Gamma \vdash x : \Gamma(x) \leadsto x \quad \text{IH\text{\text{\text{\text{\text{\text{\text{\text{\text{\text{Head}}}}}}}}}}}
\]

4.2.2 Case \(e = \lambda x.e'\)

Similar to case \(e = x\) above.

4.2.3 Case \(e = \Lambda X.e'\)

Similar to case \(e = x\) above.

\[\Box\]

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4.3 Complete \( \vdash^I \) wrt \( \vdash^F \) (TApp)

**Thm.** Under the given qualifications, if \( e \) occurs as an applicand \( e_P \) and
\[ e = e' [S], \quad t = t' [S] \]
with \( e' \neq e''[S] \),
and \( \Gamma \vdash^F e : T' \)
then \( \Gamma \vdash^I t : T \rightsquigarrow p \Rightarrow \sigma_\emptyset \)
with some \( \sigma \) such that \( \text{dom}(\sigma) = MV(\Gamma, p) \) and \( \sigma(p, T) = (e, T') \)

**Pf.** By induction on \( [S] \)

4.3.1 Case \( [S] = \emptyset \)

Our assumed derivation is
\[ \Gamma \vdash^F e : T' \]

By assumption, \( e \neq e'[S] \), for any \( e' \) and \( S \). We therefore have either \( e = e' e'' \) for some \( e' \) and \( e'' \), or else for all \( e', e'' \), and \( S \) we have \( e \neq e'[S] \) and \( e \neq e' e'' \). In both either case we can appeal to mutual induction on the completeness of \( \vdash^I \) wrt \( \vdash^F \) (either the Head or App case), and get

\[ \Gamma \vdash^I t : T \rightsquigarrow p \Rightarrow \sigma_\emptyset \]
with \( \sigma \) such that
- \( \text{dom}(\sigma) = MV(\Gamma, p) \)
- \( \sigma(p, T) = (e, T') \)

(noting that in case Head, \( \sigma = \sigma_\emptyset \))

This is exactly what we need to show, so we conclude.

4.3.2 Case \( [S] = [S'] [S] \)

Our assumed derivation is
\[ \begin{array}{c}
\Gamma \vdash^F e : [S'] : \forall X. T'' \\
\hline
\Gamma \vdash^F e : [S'] [S] : [S/X]T'' \end{array} \]
\( \text{TApp} \)

By the IH we have
\[ \Gamma \vdash^I t : [S'] : \forall X. T'' \rightsquigarrow p \Rightarrow \sigma_\emptyset \]

IH

with \( \sigma \) such that
- \( \text{dom}(\sigma) = MV(\Gamma, p) \)
- \( \sigma(p, T) = (e [S'], \forall X. T'') \)

By qualification \( \#4 \) we have \( T = \forall X. T'' \). By combining this with the post-conditions from the IH we get \( \sigma(T'') = T' \). We derive

\[ \begin{array}{c}
\Gamma \vdash^I t : [S'] : \forall X. T'' \rightsquigarrow p \Rightarrow \sigma_\emptyset \\
\hline
\Gamma \vdash^I t : [S'] [S] : [S/X]T'' \rightsquigarrow p[S] \Rightarrow \sigma_\emptyset \end{array} \]
\( \text{ITApp} \)

and note we can produce \( \sigma \), since
- \( \text{dom}(\sigma) = MV(\Gamma, p[S]) = MV(\Gamma, p) \)
- \( \sigma(p[S], [S/X]T'') = (e [S'], [S/X]T'') \)

(\( \text{dom}(\sigma) \cap X = \emptyset \) and \( \text{cod}(\sigma) \) is only those types well-formed under \( \Gamma \))
4.4 Complete \( \vdash \) wrt \( \vdash^F \) (\( App \))

Thm. Under the above qualifications, if \( \Gamma \vdash^F e \ e' : T' \) and \( t t' \) is an erasure of \( e e' \) then \( \Gamma \vdash^I t t' : T \rightsquigarrow p' \Rightarrow \sigma \) with some \( \sigma \) such that
\[
\text{dom}(\sigma) = MV(\Gamma, p') \quad \text{and} \quad \sigma(p', T) = (e e', T')
\]

Pf. Directly

Our assumed derivation is
\[
\Gamma \vdash^F e : S' \rightarrow T' \quad \Gamma \vdash^F e' : S' \quad \text{App}
\]

We can rewrite \( e = e'' [\overline{U}] \), “bubbling-out” all outermost type applications in \( e \). Since \( e \) is in an application position, we know that its erasure \( t \) may have had some number of the right-most type applications erased - so \( t = t'' [\overline{U}_1] \) where \( [\overline{U}] = [\overline{U}_1] [\overline{U}_2] \)

We examine the first premise of our assumed derivation. We now know it must have the following form:

\[
\Gamma \vdash^F e'' [\overline{U}_1] : \forall \overline{X}.T_{\overline{X}}' \quad T_{App}... \quad \Gamma \vdash^F e' : S' \quad \text{App}
\]

where the (left-to-right ordered) substitution \( [\overline{U}_2/\overline{X}]T_{\overline{X}}' = S' \rightarrow T' \)

We appeal to mutual induction on the completeness of \( \vdash^I \) wrt \( \vdash^F \) (case \( T_{App} \)) to get:

\[
\Gamma \vdash^I t'' [\overline{U}_1] : T_{\overline{X}} \rightsquigarrow p \Rightarrow \sigma \quad \text{with} \quad \sigma \quad \text{such that}
\]

\begin{itemize}
  \item \( \text{dom}(\sigma) = MV(\Gamma, p) \)
  \item \( \sigma(p, T_{\overline{X}}) = (e'' [\overline{U}_1], \forall \overline{X}.T_{\overline{X}}) \)
\end{itemize}

Now, by qualification \#3, from our \( \vdash^I \) derivation of applicand \( t \) we have that \( T_{\overline{X}} = \forall \overline{X}.S \rightarrow T \) for some \( S \) and \( T \). The use of the same bound type variables \( \overline{X} \) as used in \( \forall \overline{X}.T_{\overline{X}} \) is justified by rewriting the equality concerning \( \sigma(T_{\overline{X}}) \) above with this new information:

\[
\sigma(p, \forall \overline{X}.S \rightarrow T) = (e'' [\overline{U}_1], \forall \overline{X}.T_{\overline{X}}')
\]

We now appeal to completeness of \( \vdash^I \) wrt \( \vdash^F \). We satisfy its preconditions:

\begin{itemize}
  \item \( \Gamma \vdash^F e'' [\overline{U}_1] [\overline{U}_2] : S' \rightarrow T' \) and \( \Gamma \vdash^I t' : S' \rightsquigarrow e' \)
  \item some \( \sigma'' \) with \( \text{dom}(\sigma'') = MV(\Gamma, p [\overline{X}]) \)
  \item \( \sigma''(p [\overline{X}], S \rightarrow T) = ((e'' [\overline{U}_1]) [\overline{U}_2], S' \rightarrow T') \)
  \item \( ((\emptyset, [\overline{X}]) = ((\emptyset, [\overline{U}_2])) \)
\end{itemize}

Note that we parenthesize \( (e'' [\overline{U}_1]) \) for clarification (or at least an \textit{an attempt} for it.) We are not providing \( \overline{U}_1 \) and \( \overline{U}_2 \) to the theorem - we are providing \( \emptyset \) and \( \overline{U}_2 \), and corresponding \( \emptyset \) and \( \overline{X} \)

The \( \sigma'' \) we provide is \( \sigma \circ [\overline{U}_2/\overline{X}] \)

Having set this up, we get the following from mutual induction:
• \( \Gamma \sigma \vdash (p, \forall X.S \rightarrow T) \cdot t' : T'' \leadsto p' \Rightarrow \sigma_\emptyset \)

• some \( \sigma' \) with \( \text{dom}(\sigma') = MV(\Gamma, p') \)
  where \( \sigma'(p', T'') = (\langle e'' [U_1] [U_2] e', T' \rangle) \)

which is what we need to conclude.
\( \Box \)

4.5 Complete \( \vdash^I \) wrt \( \vdash^F \)

**Thm.** Under the above qualifications, if

• \( \Gamma \vdash^F e [U_1] [U_2] : S' \rightarrow T' \) and \( \Gamma \vdash_\emptyset t' : S' \leadsto e' \)

• and some \( \sigma \) with \( \text{dom}(\sigma) = MV(\Gamma, p) \)
  where \( \sigma(p [X_1] [X_2], S \rightarrow T) = (\langle e [U_1] [U_2], S' \rightarrow T' \rangle) \)
  (and \( (|X_1|, |X_2|) = (|U_1|, |U_2|) \))

then

• \( \Gamma; \sigma_\emptyset \vdash^I (p [X_1] [X_2], \forall [X_2], S \rightarrow T) \cdot t' : T'' \leadsto p' \Rightarrow \sigma_\emptyset \)

• with some \( \sigma' \) with \( \text{dom}(\sigma') = MV(\Gamma, p') \) and where \( \sigma(p', T'') = (e [U_1] [U_2] e', T') \)

**Pf.** By induction on \( |X_2| \)
4.5.1 Case $[X_2] = X, X_2^T$

We have $\Gamma \vdash^F e \left[ U_1 \right] [U] \left[ U_2' \right] : S' \rightarrow T'$
and $\sigma(p \left[ X_1 \right] [X] \left[ X_2^T \right], S \rightarrow T) = (e \left[ U_1 \right] [U] \left[ U_2' \right], S' \rightarrow T')$

We appeal to the IH using variable groups $X_1 = \overline{X}_1, X$ and $X_2^T$, and noting that this regrouping does not keep us from providing our given assumptions for this invocation of the IH. We get.

- $\Gamma; \sigma; \vdash^{I} (p \left[ X_1 \right] [X], \forall X_2^T S \rightarrow T) \cdot t' : T'' \sim p' \Rightarrow \sigma;$
- with some $\sigma'$ with $dom(\sigma') = MV(\Gamma, p')$ and where $\sigma(p', T'') = (e \left[ U_1 \right] [U] \left[ U_2' \right] e', T')$

From this we derive

$$\begin{array}{c}
X \in \{X, S\}, \ \Gamma \vdash S \ with \ \Gamma; \sigma; \vdash^{I} (p \left[ X_1 \right] [X], \forall X_2^T S \rightarrow T) \cdot t' : T'' \sim p' \Rightarrow \sigma \ \forall I
\end{array}$$

And provide the $\sigma'$ prime we received from our IH, noting that the conditions on it are precisely what we need to conclude.

4.5.2 Case $[X_2] = \emptyset$

We have $\Gamma \vdash^F e \left[ U_1 \right] : S' \rightarrow T'$, $\Gamma \vdash^F t' : S' \sim e'$
and $\sigma(p \left[ X_1 \right] [S \rightarrow T) = (e \left[ U_1 \right] [U] \left[ U_2' \right], S' \rightarrow T')$.

To proceed, we must do case analysis on whether $MV(\Gamma, S) = \emptyset$ or not.

**Subcase** $MV(\Gamma, S) = \emptyset$

Because $MV(\Gamma, S) \subseteq MV(\Gamma, p)$ and $\sigma(S) = (\sigma \cap MV(\Gamma, S))(S) = \sigma(S) = S'$.

So we have

$$\Gamma \vdash^F t' : S \sim e'$$

By lemma 7.5 we can derive $\Gamma \vdash^F t' : S \sim e'$ to get

$$\Gamma; \sigma; \vdash^{I} (p \left[ U_1 \right] [S \rightarrow T) \cdot t' : T \sim p \left[ U_1 \right] e' \Rightarrow \sigma \ \forall I$$

We must now provide a suitable $\sigma'$. Pick our assumed $\sigma$. Then we have

- $dom(\sigma) = MV(\Gamma, p \left[ X_1 \right] e')$
- $\sigma(p \left[ X_1 \right] e', T) = (e \left[ U_1 \right] e', T')$

allowing us to conclude this sub-case.

**Subcase** $MV(\Gamma, S) = Y \neq \emptyset$

We know that $MV(\Gamma, S) = Y \subseteq MV(\Gamma, p) = dom(\sigma)$. Let $\sigma_Y = \sigma \cap Y$. Then we know $\sigma(S) = \sigma_Y(S) = S'$.

We have

$$\Gamma \vdash^F t' : \sigma_Y(S) \sim e'$$

We can derive
\[ MV(\Gamma, S) = Y \neq \emptyset \quad \Gamma \vdash \sigma \vdash t' : \sigma(S) \leadsto e \quad \Gamma; \sigma \vdash ! (p \ [X_1], S \to T) \cdot t' : \sigma(S) \leadsto \sigma(p \ [X_1]) e' \Rightarrow \sigma \vdash I \]

We must now pick a suitable \( \sigma' \). Pick \( \sigma - \sigma(S) \). We have

- \( \text{dom}(\sigma - \sigma(S)) = MV(\Gamma, p \ [X_1]) e') - Y = MV(\Gamma, \sigma(S) p \ [X_1]) e') \)
- \( \sigma'(\sigma(S) p \ [X_1]) e', \sigma(S) p \ [X_1] e' \)

\[ = \sigma(p \ [X_1] e', T) \]
\[ = (e \ [U_1] e', T') \]

which is what we need to conclude.

\[ \square \]

5 Lemmas: Soundness of \( \vdash D \) wrt \( \vdash F \)

5.1 Sound \( \sigma \) wrt \( \vdash F \)

Thm. If \( \Gamma, X \vdash F p : T \) where for all \( X \in X \), \( X \) occurs only in the type applications of the outermost spine of \( p \)

and \( \text{dom}(\sigma) \subseteq X \) and for all \( S \in \text{cod}(\sigma), \Gamma \vdash S \ w f \)

then \( \Gamma, (X - \text{dom}(\sigma)) \vdash F \sigma(p) : \sigma(T) \)

Pf. By induction on the assumed derivation.

5.1.1 Case Var:

Our assumed derivation is

\[ \Gamma, X \vdash F x : (\Gamma, X)(x) \quad Var \]

We note that since \( x \) is a term variable, \( x \notin X \). Let \( X' = X - \text{cod}(\sigma) \). We can derive

\[ \Gamma \vdash F x : \Gamma(x) \quad Var \]
\[ \Gamma, X' \vdash F x : (\Gamma, X')(x) \quad Weaken \]

5.1.2 Case Abs:

Our assumed derivation is

\[ \Gamma, X, x : T \vdash F e : S \quad Abs \]
\[ \Gamma, X' \vdash F \lambda x : T. e : T \to S \]

By definition \( \sigma(\lambda x : T. e) = \lambda x : T. e \). By assumption \( X \cap \text{FV}(\lambda x : T. e) = \emptyset \), and therefore \( X \cap \text{FV}(S \to T) = \emptyset \). So, let \( X' = X - \text{dom}(\sigma) \), and we can therefore conclude with

\[ \Gamma, x : T \vdash F e : T \quad Abs \]
\[ \Gamma, X' \vdash F \lambda x : T. e : S \to T \quad Weaken \]

nothing that from our reasoning above, \( \sigma(\lambda x : T. e) = \lambda x : T. e \) and \( \sigma(S \to T) = S \to T \)
5.1.3 Case \( TAbs \):

Our assumed derivation is:

\[
\frac{\Gamma, X, X \vdash F : T}{\Gamma, X \vdash \lambda X. e : \forall X. T} \quad \text{\( TAbs \)}
\]

The proof is handled similarly as above, noting that bound type variable \( X \notin \text{dom}(\sigma) \)

5.1.4 Case \( TApp \):

Our assumed derivation is:

\[
\frac{\Gamma, X \vdash F : \forall X. T}{\Gamma, X \vdash F[e[R] : [R/X]T]} \quad \text{\( TApp \)}
\]

Let \( \overline{X} = \overline{X} - \text{dom}(\sigma) \). By the IH on the first premise, we have

\[
\frac{\Gamma, X \vdash F : \forall X. T}{\Gamma, \overline{X} \vdash \sigma(e) : \forall X. \sigma(T)} \quad \text{IH}
\]

Now, by assumption either \( R \in \overline{X} \) or else \( \Gamma \vdash R \text{ wf} \). If the former, we must ask whether \( R \in \text{dom}(\sigma) \). If it is, let \( \sigma(R) = S \), where by assumption we have \( \Gamma \vdash S \text{ wf} \) and by weakening \( \Gamma, \overline{X}' \vdash S \text{ wf} \). Then we can derive

\[
\frac{\Gamma, \overline{X} \vdash \sigma(p[R])) : \sigma(T)}{\Gamma \vdash F e' : S}
\]

If \( R \in \overline{X} \) but \( R \notin \text{dom}(\sigma) \), we have that \( R \in \overline{X}' \). We can derive

\[
\frac{\Gamma, \overline{X} \vdash \sigma(p)[R] : \sigma([R/X]T).}{\Gamma, \overline{X} \vdash \sigma(p)[R] : \sigma([R/X]T).}
\]

Similarly, if \( R \notin \overline{X} \) then by the implicit premise of our assumed derivation we have \( \Gamma \vdash R \text{ wf} \), and we can derive

\[
\frac{\Gamma, \overline{X} \vdash \sigma(p)[R] : \sigma([R/X]T).}{\Gamma, \overline{X} \vdash \sigma(p)[R] : \sigma([R/X]T).}
\]

In each of these sub-cases, we can derive what we need.

5.1.5 Case \( \text{App} \):

Our assumed derivation is

\[
\frac{\Gamma, X \vdash F : S \rightarrow T \quad \Gamma, X \vdash e' : S}{\Gamma, X \vdash e : S \rightarrow T} \quad \text{\( \text{App} \)}
\]

Let \( \overline{X} = \overline{X} - \text{dom}(\sigma) \). By the IH on the first premise, we have

\[
\frac{\Gamma, X \vdash F : S \rightarrow T}{\Gamma, \overline{X} \vdash \sigma(e) : \sigma(S \rightarrow T)} \quad \text{IH}
\]

By assumption, \( FV(e') \cap \overline{X} = \emptyset \), and therefore \( FV(S) \cap \overline{X} = \emptyset, \sigma(S) = S \) and \( \sigma(e') = e' \). We can therefore derive

\[
\frac{\Gamma \vdash F' : S \quad \Gamma, \overline{X} \vdash e' : S}{\Gamma, \overline{X} \vdash e' : S} \quad \text{Weaken}
\]

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and conclude
\[ \Gamma, \overline{X} \vdash^F \sigma(e) : S \rightarrow \sigma(T) \quad \Gamma, \overline{X} \vdash^F \sigma(e') : S \]
\[ \frac{}{\Gamma, \overline{X} \vdash^F \sigma(e e') : \sigma(T)} \quad \text{App} \]
\[ \Box \]

5.2 Lemma: \( \sigma \) introduces no meta-variables

Lemma 1.

- If \( \Gamma \vdash t : T \rightsquigarrow (p, \sigma) \) then \( \text{dom}(\sigma) \cap \text{BTV}(\text{cod}(\sigma)) = \emptyset \)
- If \( \Gamma \vdash t : T \rightsquigarrow (p', \sigma') \) and \( \text{dom}(\sigma) \cap \text{BTV}(\text{cod}(\sigma)) = \emptyset \) then \( \text{dom}(\sigma') \cap \text{BTV}(\text{cod}(\sigma')) = \emptyset \)

Pf. By induction on the assumed derivations, noting that only rule \( \cdot \forall \) adds any meta-variable solutions and that these are generated from a type that does not have access to solutions in the input substitution \( \sigma \).

6 Lemmas: Soundness of \( \vdash^A \) wrt \( \vdash^D \)

6.1 Lemma: Sound \( \lceil \rceil \) wrt \( \vdash^A \)

Thm. If \( \overline{X} \vdash T := P \Rightarrow (\sigma, W) \) then \( \lceil W \rceil = T \)

Pf. A straightforward induction on the assumed derivation.

6.2 Lemma: Sound \( \sigma \) wrt \( \vdash^A \) (1)

Thm. If \( \overline{X}, X \vdash T := P \Rightarrow (\sigma, W) \)

and \( [S/X]W \) is well-formed,

then \( \overline{X} \vdash [S/X]T := P \Rightarrow (\sigma - [S/X], [S/X]W) \)

Pf. By induction on the assumed derivation.

Note that for \( [S/X]W \) to be well-formed, we may need to witness another internal derivation of \( \vdash^A \) while performing the substitution. This makes the result straightforward.

6.3 Lemma: Sound \( \sigma \) wrt \( \vdash^A \) (2)

Thm. If \( \overline{X} \vdash T := P \Rightarrow (\sigma \circ \sigma', W) \)

then \( \overline{X} - \text{dom}(\sigma') \vdash^A \sigma'(T) := P(\sigma, \sigma'(W)) \)

Pf. By an easy inductive argument on the assumed derivation.

6.4 Lemma: Sound \( \vdash^A \) wrt \( \vdash^A \)

Thm. If \( \Gamma; P \vdash^A t : W \rightsquigarrow p \Rightarrow \sigma \) then \( \text{MV}(\Gamma, p) \vdash^A \lceil W \rceil := P \Rightarrow (\sigma, W) \)

Pf. By induction on the assumed derivation

6.4.1 Case "Head"

Our assumed derivation is
\[ t \neq t[S] \wedge t \neq t' \quad \frac{}{\Gamma \vdash^A t : T \rightsquigarrow e} \quad \frac{}{\emptyset \vdash^A T := ? \Rightarrow P \Rightarrow (\sigma_{\emptyset}, W)} \]
\[ \frac{}{\Gamma; ? \Rightarrow P \vdash^A t : W \rightsquigarrow e \Rightarrow \sigma_{\emptyset}} \quad \text{?Head} \]

We apply lemma 6.1 on the third hypothesis to get
\[ MV(\Gamma, e) = \emptyset \vdash [W] := ? \rightarrow P \Rightarrow (\sigma_\emptyset, W) \]

which is what we need.

6.4.2 Case ?TApp

Our assumed derivation is

\[
\Gamma; ? \Rightarrow P \vdash^\delta t : \forall X = R.W \leadsto p \Rightarrow \sigma \quad R = X \lor R = S
\]

\[ \Gamma; ? \Rightarrow P \vdash^{A^?} t[S] : [S/X]W \leadsto p[S] \Rightarrow \sigma \]

\(?TApp\)

We invoke the IH on the first premise, yielding

\[ MV(\Gamma, p) \vdash \forall X. [W] := ? \rightarrow P \Rightarrow (\sigma, W) \]

The only rule which allows us to form this conclusion is \(\vdash^{A^?}\), with premise

\[ MV(\Gamma, p), X \vdash [W] := ? \rightarrow P \Rightarrow (\sigma \circ [R/X], W) \]

Because \([S/X]W\) is (implicitly) well-formed, by lemma 6.2 we have

\[ MV(\Gamma, p[S]) \vdash [[S/X]W] := ? \rightarrow P \Rightarrow (\sigma, [S/X]W) \]

allowing us to complete the proof.

6.4.3 Case ?App

Our assumed derivation is

\[
\Gamma; ? \Rightarrow P \vdash^\delta t : W \leadsto p \Rightarrow \sigma \quad \Gamma; \sigma \vdash^A (p, W) \cdot t' : W' \leadsto p' \Rightarrow \sigma'
\]

\[ \Gamma; P \vdash^{A^?} t' : W' \leadsto p' \Rightarrow \sigma' \]

\(?App\)

By the IH on the first premise, we have

\[ MV(\Gamma, p) \vdash [W] := ? \rightarrow P \Rightarrow (\sigma, W) \]

With this and with the second premise, we appeal to lemma 6.5 to conclude

\[ MV(\Gamma, p') \vdash [W'] := P \Rightarrow (\sigma, W') \]

6.5 Lemma: Sound \(\vdash^{A^?}\) wrt \(\vdash=\)

Thm. If \( MV(\Gamma, p) \vdash [W] := ? \rightarrow P \Rightarrow (\sigma, W) \)
and \( \Gamma; \sigma \vdash^A (p, W) \cdot t' : W' \leadsto p' \Rightarrow \sigma' \)
then \( MV(\Gamma, p') \vdash [W'] := P \Rightarrow (\sigma, W') \)

Pf. By induction on the assumed derivation of \(\vdash^A\).

6.5.1 Case \(\forall\)

Our assumed derivation is

\[
\Gamma; \sigma \circ [R/X] \vdash^A (p[X], W) \cdot t' : W' \leadsto p' \Rightarrow \sigma'
\]

\[ \Gamma; \sigma \vdash^A (p, \forall X = R.W) \cdot t' : W' \leadsto p' \Rightarrow \sigma' \]

\(\forall\)
and our assumed match is

\[ MV(\Gamma, p) \vdash_:= A.X_\boxed{\boxed{W}} := ? \rightarrow P \Rightarrow (\sigma, \forall X = R.W) \]

The only rules giving us this match is := \forall, with premise

\[ MV(\Gamma, p[X]) \vdash_:= \boxed{W} := ? \rightarrow P \Rightarrow (\sigma \circ [R/X], W) \]

With this and with the premise of the derivation, we can invoke the IH to conclude

\[ MV(\Gamma, p') \vdash_:= \boxed{W'} := P \Rightarrow (\sigma', W') \]

6.5.2 Case \cdot \downarrow

Our assumed derivation is

\[ MV(\Gamma, \sigma(S)) = \emptyset \quad \Gamma \vdash_A t' : S \rightsquigarrow e \]

\[ \Gamma; \sigma \vdash_A (p, S \Rightarrow \boxed{W}) \cdot t': \boxed{W} \rightsquigarrow p e' \Rightarrow \sigma \cdot \downarrow \]

and our assumed match is

\[ MV(\Gamma, p) \vdash_:= S \rightarrow \boxed{W} := ? \rightarrow P \Rightarrow (\sigma, W) \]

The only rule allowing us to form this match is := \rightarrow, with premise

\[ MV(\Gamma, p e') \vdash_:= \boxed{W} := P \Rightarrow (\sigma, W) \]

(since \( MV(\Gamma, e') = \emptyset \)), allowing us to conclude.

6.5.3 Case \cdot \uparrow

Our assumed derivation is

\[ MV(\Gamma, \sigma(S)) = Y \neq \emptyset \quad \Gamma \vdash_A t' : S \rightsquigarrow e \]

\[
\Gamma; \sigma \vdash_A (p, S \Rightarrow \boxed{W}) \cdot t': \boxed{W} \rightsquigarrow \boxed{U/Y} p e' \Rightarrow \sigma \cdot \uparrow
\]

and our assumed match is

\[ MV(\Gamma, p) \vdash_:= S \rightarrow \boxed{W} := ? \rightarrow P \Rightarrow (\sigma, W) \]

The only rule allowing us to form this match is := \rightarrow, with premise

\[ MV(\Gamma, p e') \vdash_:= \boxed{W} := P \Rightarrow (\sigma, W) \]

By lemma 6.2 and by noting that \( Y \cap \text{dom}(\sigma) = \emptyset \) from our first premise, we have

\[ MV(\Gamma, \boxed{U/Y} p e') \vdash_:= \boxed{U/Y} W := P \Rightarrow (\sigma, \boxed{U/Y} W) \]

which allows us to conclude.

\[ \square \]

6.6 Lemma: Sound \vdash_:= \text{ wrt } \vdash_{wf}

Lm. If \( X \vdash_:= T := P \Rightarrow (\sigma, W) \)

with \( \Gamma \vdash P \text{ wf} \) and \( \Gamma, X \vdash T \text{ wf} \)
then $\forall X \in X, \sigma(X) = X$ or $\Gamma \vdash \sigma(S) \, w.f$

Pf. By an simple inductive argument on the assumed derivation.

7 Lemmas: Completeness of $\vdash^A$ wrt $\vdash^D$

7.1 Lemma: $\vdash^I$ invertible $\vdash^=\cdot$

Lemma If $\Gamma \vdash^I (p:T, \sigma) \cdot t': T' \rightsquigarrow p' \Rightarrow \sigma'$
and $MV(\Gamma, p') \vdash^= T' := P \Rightarrow (\sigma', W'), \sigma' \subseteq \sigma'^A$
then exists $(\sigma^A, W)$ where $MV(\Gamma, p) \vdash^= T := \Rightarrow \sigma^A, W), \sigma \subseteq \sigma^A$

Pf. By induction on the assumed derivation

7.1.1 Case $\forall$

Our assumed derivation is

$$
R \in \{X, S\}, \Gamma \vdash S \, w.f \quad \Gamma; \sigma \circ [R/X] \vdash^I (p[X], T) \cdot t': T' \rightsquigarrow p' \Rightarrow \sigma' \quad \forall M
$$

Our assumed match is

$$MV(\Gamma, p') \vdash^= T' := P \Rightarrow (\sigma'^A, W'), \sigma' \subseteq \sigma'^A$$

We invoke the IH on this match and the second premise to get

$$MV(\Gamma, p[X]) \vdash^= T := \Rightarrow \sigma^A, W), \sigma \circ [R/X] \subseteq \sigma^A$$

Applying matching rule $\vdash^= \Rightarrow \sigma^A$, $\forall X = R.W)

7.1.2 Case $\downarrow$

Our assumed derivation is

$$MV(\Gamma, \sigma(S)) = \emptyset \quad \Gamma \vdash \downarrow t' : \sigma(S) \rightsquigarrow e' \quad \downarrow M
$$

Our assumed match is

$$MV(\Gamma, p e') \vdash^= T := P \Rightarrow (\sigma'^A, W), \sigma \subseteq \sigma'^A.$$

We invoke matching rule $\Rightarrow$ to conclude (noting $MV(\Gamma, e') = \emptyset$)

$$MV(\Gamma, p) \vdash^= S \Rightarrow T := ? \Rightarrow P \Rightarrow (\sigma'^A, S \Rightarrow W), \sigma \subseteq \sigma'^A$$

7.1.3 Case $\uparrow$

Our assumed derivation is

$$MV(\Gamma, \sigma(S)) = Y \neq \emptyset \quad \Gamma \vdash \uparrow t' : [U/Y] \circ \sigma(S) \rightsquigarrow e \quad \uparrow M
$$

$$\Gamma; \sigma \vdash^I (p, S \Rightarrow T) \cdot t' : [U/Y]T \rightsquigarrow [U/Y]p e' \Rightarrow \sigma$$
Our assumed match is

\[ MV(\Gamma, [U/Y]p e') ::= [U/Y]T ::= P \Rightarrow (\sigma'^A, W') \]

By lemma 7.2 on the invertibility of substitutions in a match, we have

\[ MV(\Gamma, p e') ::= T ::= P \Rightarrow (\sigma'^A \circ \sigma_T, W'_T), \sigma_T \subseteq [U/Y] \]

Noting that \( \sigma \subseteq \sigma'^A \circ \sigma_T \), we apply rule \( \Rightarrow \) to conclude

\[ MV(\Gamma, p e') ::= S \Rightarrow T ::= P \Rightarrow (\sigma'^A \circ \sigma_T, S \Rightarrow W'_T) \]

7.2 Lemma \( \vdash ::= \) invertible \( \sigma \)

Lemma

If \( \Gamma \vdash ::= [U/Y]T ::= P \Rightarrow (\sigma, W) \) and \( \Gamma \cap FV(U) = \emptyset \) then \( \Gamma, Y \vdash ::= T ::= P \Rightarrow (\sigma \circ \sigma_T, W_T) \) where

- \( \sigma_T \subseteq [U/Y] \)
- \( [U/Y]W_T = W \)

7.3 Lemma \( \vdash ::= \) unique

Lemma

If \( \Gamma \vdash ::= T ::= P \Rightarrow (\sigma, W) \) and \( \Gamma \vdash ::= T ::= P \Rightarrow (\sigma', W') \), then \( (\sigma, W) = (\sigma', W') \)

7.4 Lemma: Complete \( \vdash_A \Rightarrow \vdash_A \)

Thm. If \( \Gamma \vdash \Rightarrow t : T \Rightarrow e \) then \( \Gamma \vdash \Rightarrow t : T \Rightarrow e \) Pf. By induction on the assumed derivation. Most cases come down to a use of the rule ChkSyn so we look closely only at AppSyn

7.4.1 Case AppSyn:

Our assumed derivation is

\[
\frac{\Gamma \vdash t \; t' : T' \Rightarrow e \Rightarrow \sigma_\emptyset \quad MV(\Gamma, e') = \emptyset}{\Gamma \vdash \Rightarrow t \; t' : T' \Rightarrow e'} \quad \text{AppSyn}
\]

From a little bit of equational reasoning we can see that

\[ MV(\Gamma, e') = MV(\Gamma, T') = dom(\sigma_\emptyset) = \emptyset \]

But this is what we need to conclude with derivation

\[
\frac{\Gamma \vdash t' \; t' : T' \Rightarrow e \Rightarrow \sigma_\emptyset \quad MV(\Gamma, e') = MV(\Gamma, T') = dom(\sigma_\emptyset) \quad \sigma_\emptyset(T') = T'}{\Gamma \vdash \Rightarrow t' \; t' : \sigma_\emptyset(e')} \quad \text{AppChk}
\]

7.5 Complete \( \vdash \emptyset \Rightarrow \vdash \emptyset \)

Thm. If \( \Gamma \vdash \emptyset t : T \Rightarrow e \) then \( \Gamma \vdash \emptyset t : T \Rightarrow e \) Pf. Take the assumed derivation of \( \vdash \emptyset \), invoke completeness of \( \vdash_A \emptyset \Rightarrow \vdash_A \), invoke completeness of \( \vdash_A \emptyset \Rightarrow \vdash_A \), and then finish with soundness of \( \vdash_A \emptyset \Rightarrow \vdash_A \emptyset \).
8 Lemmas: Qualified Completeness of $\vdash_{\uparrow}$ wrt $\vdash_F$

8.1 Sound $\sigma$ wrt $\vdash_I$

Thm.

- If $\Gamma \vdash_I t : T \leadsto (p, \sigma)$ then $\text{dom}(\sigma) \subseteq \text{MV}(\Gamma, p)$
- If $\Gamma \vdash_I (p: T, \sigma) \cdot t' : T' \leadsto (p', \sigma')$
  and $\text{dom}(\sigma) \subseteq \text{MV}(\Gamma, p)$
  then $\text{dom}(\sigma') \subseteq \text{MV}(\Gamma, p')$

Pf. By a mutual induction on the assumed premise in both cases.

9 Lemmas: Misc.

9.1 Matching Arrows of $P$ and $W$

Theorem 4. Let $\text{arr}_P(P)$ be the number of prototype arrows prefixing $P$, and $\text{arr}_W(W)$ the number of decorated-type arrows preceding $W$.

- If $\Gamma; P \vdash_A^\uparrow t : W \leadsto (p, \sigma)$ then $\text{arr}_W(W) \leq \text{arr}_P(P)$
- If $\Gamma \vdash_A (p: W, \sigma) \cdot t' : W' \leadsto (p', \sigma')$ then $\text{arr}_W(W') = \text{arr}_W(W) - 1$
- If $\bar{\Gamma} \vdash_{\Rightarrow} T := P \Rightarrow (\sigma, W)$ then $\text{arr}_P(P) = \text{arr}_W(W)$

Proof. TODO