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SOLVING THE NONLINEAR POISSON EQUATION ON THE UNIT DISK

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ABSTRACT. We propose and analyze a numerical method for solving the nonlinear Poisson equation $-\Delta u = f(\cdot, u)$ on the unit disk with zero Dirichlet boundary conditions. The problem is reformulated as a nonlinear integral equation. We use a Galerkin method with polynomials as approximations. The speed of convergence is shown to be very rapid; and experimentally the maximum error is exponentially decreasing when it is regarded as a function of the degree of the approximating polynomial.

1. Introduction. In the earlier papers [**2, 4**] a Galerkin method was proposed, analyzed, and illustrated for the numerical solution of a Dirichlet problem for a semi-linear elliptic boundary value problem of the form

$$\begin{aligned} -\Delta U &= F(\cdot, U) \quad \text{on} \quad \Omega, \\ U &= G \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{1}$$

In this, $\Omega \subset \mathbf{R}^2$ is a simply-connected open domain with a boundary $\partial\Omega$. It was assumed that there is a known conformal mapping from a standard open domain D to Ω, and then the problem (1) was reduced to an equivalent problem on D,

$$\begin{aligned} -\Delta u &= f(\cdot, u) \quad \text{on} \quad D, \\ u &= g \quad \text{on} \quad \partial D. \end{aligned} \tag{2}$$

This equation was then converted to an equivalent, but nonstandard, integral equation over D. A Galerkin method was used to solve the integral equation, with the eigenfunctions of the Laplacian operator on the standard domain D as the basis functions. The method was simple to program and relatively inexpensive, but it converged slowly with respect to the dimension of the approximation space being used.

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In this paper assume D is the unit disk in $\mathbf{R}^2$; and assume further that g satisfies the homogeneous boundary condition, $g(x) \equiv 0$, on ∂D. For this situation we give a numerical method that converges much more rapidly than the earlier method described above. As in the earlier papers [**2, 4**], use a Galerkin method; but now use polynomials as the approximating functions. The mathematical reformulation we use for (2) and the numerical method for solving it are described in Section 2. A theoretical error and convergence analysis is given in Section 3, and an illustrative numerical example is given in Section 4. The paper concludes in Section 5 by introducing another basis for the polynomials over D, a basis that has improved stability properties when compared with the basis used in Sections 2 and 4.

For nonhomogeneous boundary conditions and for conformal transformations of the unit disk, the reader is referred to the earlier paper [**2**]. References to earlier work on the numerical solution of (1) can be found in the bibliographies of [**2, 4**].

2. Preliminaries. Let $G(x, y; \xi, \eta)$ be the Green's function for the problem

$$\begin{aligned} -\Delta u &= \psi \quad \text{in} \quad D, \\ u &= 0 \quad \text{on} \quad \partial D, \end{aligned}$$

assuming that ψ is known. Using this Green's function, the solution u to (2) satisfies

$$u(x,y) = \int_D G(x,y,\xi,\eta)\, f(\xi,\eta,u(\xi,\eta))\, d\xi\, d\eta, \quad (x,y) \in \overline{D}. \tag{3}$$

As in Kumar and Sloan [**8**], we introduce $v(x,y) = f(x,y,u(x,y))$. The function v is a solution of

$$v(x,y) = f\left(x, y, \int_D G(x,y,\xi,\eta)\, v(\xi,\eta)\, d\xi\, d\eta\right), \quad (x,y) \in \overline{D}. \tag{4}$$

This is the equation that we solve using Galerkin's method. After finding $v(x,y)$, we can calculate

$$u(x,y) = \int_D G(x,y,\xi,\eta)\, v(\xi,\eta)\, d\xi\, d\eta, \quad (x,y) \in \overline{D}. \tag{5}$$

This is discussed in more detail later in the paper.

2.1 *Galerkin's method.* Let Π_d denote the polynomials in (x, y) of degree $\leq d$, with d a nonnegative integer. The dimension of Π_d is

$$N \equiv N_d = \frac{1}{2}(d+1)(d+2).$$

Let $\{\Lambda_n(x, y) : 1 \leq n \leq N\}$ be a basis for Π_d. Then Galerkin's method for solving (4) with Π_d as the approximating space is as follows. Find

$$v_d(\xi, \eta) = \sum_{m=1}^{N} \alpha_m \Lambda_m(x, y) \tag{6}$$

such that $v_d(\xi, \eta) \approx v(\xi, \eta)$. Determine the coefficients $\{\alpha_m\}$ by solving the nonlinear system

$$\sum_{m=1}^{N} \alpha_m (\Lambda_m, \Lambda_n) - \left(f\left(\cdot, \cdot, \int_D G(\cdot, \cdot, \xi, \eta) v_d(\xi, \eta)\, d\xi d\eta \right), \Lambda_n \right) = 0 \tag{7}$$

for $n = 1, \ldots, N$. An approximation of the solution of the original problem (3) is defined, using (5), as

$$u_d(x, y) = \int_D G(x, y, \xi, \eta)\, v_d(\xi, \eta)\, d\xi\, d\eta, \quad (x, y) \in \overline{D}. \tag{8}$$

A major advantage of solving (4) as compared to solving (3) can be seen from this formula. Begin by noting that the system (7) must be solved by iteration, e.g., using Newton's method or Broyden's method, obtaining a sequence of iterates

$$v_d^{(k)}(x, y) = \sum_{m=1}^{N} \alpha_m^{(k)} \Lambda_m(x, y), \qquad k = 0, 1, \ldots .$$

With each iterate, we must calculate the integrals

$$\int G(x, y, \xi, \eta)\, v_d^{(k)}(\xi, \eta)\, d\xi\, d\eta = \sum_{m=1}^{N} \alpha_m^{(k)} \int_D G(x, y, \xi, \eta)\, \Lambda_m(\xi, \eta)\, d\xi\, d\eta, \quad (x, y) \in \overline{D}. \tag{9}$$

The integrals on the right side do not depend on k, and thus they need to be calculated only once. In contrast, consider using Galerkin's method to solve formula (3). If we apply an iterative method of solution and obtain a sequence $\left\{u_d^{(k)}(\xi,\eta) : k = 0, 1, \dots\right\}$, then this will require calculation of the integral term

$$\int_D G(x,y,\xi,\eta)\, f(\xi,\eta,u_d^{(k)}(\xi,\eta))\, d\xi\, d\eta$$

for each iterate $u_d^{(k)}$. The lack of linearity with regard to $u_d^{(k)}$ means that this integral must be recalculated for each value of k, a considerable increase in computational cost. The improvement in calculational cost was the primary reason motivating Kumar and Sloan [**8**] in proposing the reformulation for Hammerstein nonlinear integral equations in the manner described above.

Another problem remains, that of evaluating the integrals in (9). These have a singular integrand due to G being singular,

$$G(P,Q) = \frac{1}{2\pi}\log\frac{|P-Q|}{|\mathcal{T}(P)-Q|}, \quad P \neq Q, \quad Q \in D, \quad P \in \overline{D}.$$

$\mathcal{T}(P)$ denotes the inverse of P with respect to the unit disk,

$$\mathcal{T}(r\cos\theta, r\sin\theta) = \frac{1}{r}(\cos\theta, \sin\theta). \quad r \leq 1$$

It would be advantageous to choose the basis functions $\{\Lambda_m(x,y) : 1 \leq m \leq N\}$ so as to avoid the need to evaluate numerically the integrals in (9).

2.2 *Choosing a polynomial basis.* Begin by considering the mapping $\mathcal{D} : \Pi_d \to \Pi_d$,

$$\Phi(x,y) \longrightarrow -\Delta\left[\left(1-x^2-y^2\right)\Phi(x,y)\right], \quad \Phi \in \Pi_d. \tag{10}$$

Trivially, the mapping $\mathcal{D} : \Pi_d \to \Pi_d$ is *into*.

Note next that this mapping is one-to-one. To see this, assume

$$-\Delta\left[\left(1-x^2-y^2\right)\Phi(x,y)\right] = 0$$

for some $\Phi \in \Pi_d$. Let $\Psi(x,y) = \left(1 - x^2 - y^2\right) \Phi(x,y)$, a polynomial of degree $\leq d+2$. Since $-\Delta\Psi = 0$, and since $\Psi(x,y) \equiv 0$ on ∂D, we have by the uniqueness of the solvability of the Dirichlet problem on D that $\Psi(x,y) \equiv 0$ on D. This then implies that $\Phi(x,y) \equiv 0$ on D.

Since the mapping is both one-to-one and into, it follows from Π_d being finite-dimensional that the mapping is *onto*. We use this to produce a special basis for Π_d.

Let $\{\Phi_n(x,y) : 1 \leq n \leq N\}$ be a basis for Π_d, and let

$$\begin{aligned} \Psi_n(x,y) &= \left(1 - x^2 - y^2\right) \Phi_n(x,y) \\ \Lambda_n(x,y) &= -\Delta \Psi_n(x,y) \end{aligned} \tag{11}$$

for $n = 1, \ldots, N$. This is the basis we use for the Galerkin method of (7). With it note that for the Green's function integrals in (9),

$$\int_D G(x,y,\xi,\eta)\, \Lambda_m(\xi,\eta)\, d\xi\, d\eta = \Psi_m(x,y), \quad m = 1, \ldots, N. \tag{12}$$

This avoids the need to do any numerical integration of these integrals, which results in an enormous savings in computational time. The nonlinear system (7) becomes

(13)

$$\sum_{m=1}^{N} \alpha_m \left(\Lambda_m, \Lambda_n\right) - \left(f\left(\cdot, \cdot, \sum_{m=1}^{N} \alpha_m \Psi_m\right), \Lambda_n \right) = 0, \quad n = 1, \ldots, N.$$

For the solution of the original problem, we combine (6), (8) and (12), leading to the definition

$$u_d(x,y) = \sum_{m=1}^{N} \alpha_m \Psi_m(x,y).$$

How do we choose the basis $\{\Phi_n(x,y)\}_{m=1}^{N}$ for Π_d? We want to have a basis for which the linearization of the system (7) is well-conditioned. To this end we have chosen $\{\Phi_n(x,y)\}_{m=1}^{N}$ to be an orthonormal basis of Π_d. For an introduction to this topic, see the important book of Dunkl and Xu [**5**]. Unlike the situation for the single variable case, there are

many possible orthonormal bases over D. We have chosen one that is particularly convenient for the computations in (11). These are the "ridge polynomials" introduced by Logan and Shepp [**9**] for solving an image reconstruction problem. We summarize here the results needed for our work.

Let

$$\mathcal{V}_d = \{P \in \Pi_d : (P, Q) = 0 \ \forall\, Q \in \Pi_{d-1}\}$$

the polynomials of degree d that are orthogonal to all elements of Π_{d-1}. Then the dimension of $\mathcal{V}_d$ is $d+1$; moreover,

$$\Pi_d = \mathcal{V}_0 \oplus \mathcal{V}_1 \oplus \cdots \oplus \mathcal{V}_d. \tag{14}$$

It is standard to construct orthonormal bases of each $\mathcal{V}_n$ and to then combine them to form an orthonormal basis of Π_d using the latter decomposition. As an orthonormal basis of $\mathcal{V}_n$, we use

$$\begin{gathered}\Phi_{n,k}(x,y) = \frac{1}{\sqrt{\pi}} U_n\left(x\cos(kh) + y\sin(kh)\right), \\ (x,y) \in D, \quad h = \frac{\pi}{n+1}\end{gathered} \tag{15}$$

for $k = 0, 1, \ldots, n$. The function U_n is the Chebyshev polynomial of the second kind of degree n:

$$\begin{gathered}U_n(t) = \frac{\sin(n+1)\,\theta}{\sin\theta}, \\ t = \cos\theta, \quad -1 \le t \le 1, \quad n = 0, 1, \ldots .\end{gathered} \tag{16}$$

The family $\{\Phi_{n,k}\}_{k=0}^{n}$ is an orthonormal basis of $\mathcal{V}_n$. As a basis of Π_d, we order $\{\Phi_{n,k}\}$ lexicographically based on the orderings in (14) and (15):

$$\{\Phi_n\}_{m=1}^{N} = \{\Phi_{0,0},\, \Phi_{1,0},\, \Phi_{1,1},\, \Phi_{2,0},\, \ldots,\, \Phi_{n,0},\, \ldots, \Phi_{n,n}\}.$$

Returning to (11), we have

$$\begin{aligned}\Psi_{n,k}(x,y) &= \left(1 - x^2 - y^2\right)\Phi_{n,k}(x,y) \\ \Lambda_{n,k}(x,y) &= -\Delta\Psi_{n,k}(x,y).\end{aligned}$$

Carrying out the actual computations using (15), we have

$$\Lambda_{n,k}(x,y) = \frac{1}{\sqrt{\pi}}\left[4U_n(t) + 4tU_n'(t) - \left(1-x^2-y^2\right)U_n''(t)\right] \tag{17}$$
$$t = x\cos(kh) + y\sin(kh)\,.$$

We evaluate $U_n(t)$, $U_n'(t)$, $U_n''(t)$ using the standard triple recursion relations

$$\begin{aligned} U_{n+1}(t) &= 2tU_n(t) - U_{n-1}(t) \\ U_{n+1}'(t) &= 2U_n(t) + 2tU_n'(t) - U_{n-1}'(t) \\ U_{n+1}''(t) &= 4U_n'(t) + 2tU_n''(t) - U_{n-1}''(t). \end{aligned}$$

To examine the possible ill-conditioning of the basis $\{\Lambda_m\}_{m=1}^N$, we give in Table 1 the condition numbers of the Gram matrix

$$\mathcal{M}_d = [(\Lambda_n, \Lambda_m)]_{n,m=1}^N\,.$$

These are increasing but are still small enough as to allow stable computations in the linearization of the nonlinear system (7).

TABLE 1. Condition numbers of Gram matrix $\mathcal{M}_d$.

d	order $(\mathcal{M}_d)$	cond $(\mathcal{M}_d)$	d	order $(\mathcal{M}_d)$	cond $(\mathcal{M}_d)$
1	3	4	11	78	2965
2	6	23	12	91	4050
3	10	54	13	105	5304
4	15	127	14	120	6946
5	21	229	15	136	8807
6	28	415	16	153	11167
7	36	655	17	171	13805
8	45	1034	18	190	17066
9	55	1498	19	210	20672
10	66	2170	20	231	25038

3. Convergence. Introduce the Nemyckii operator

$$(\mathcal{F}(u))(x,y) = f(x,y,u(x,y)), \tag{18}$$

and the linear integral operator

$$(\mathcal{G}v)(x,y) = \int_D G(x,y;\xi,\eta)\, v(\xi,\eta)\, d\xi\, d\eta. \tag{19}$$

The equations (3) and (4) are written symbolically as

$$u = \mathcal{G}\mathcal{F}(u) \tag{20}$$

$$v = \mathcal{F}(u)$$

$$v = \mathcal{F}(\mathcal{G}v). \tag{21}$$

Let $\mathcal{P}_d$ be the $L^2(D)$ orthogonal projection of $L^2(D)$ onto Π_d. Then the Galerkin solution $v_d \in \Pi_d$ of (6) and (7) satisfies the operator equation

$$v_d = \mathcal{P}_d\mathcal{F}(\mathcal{G}v_d). \tag{22}$$

Also, (8) is written symbolically as

$$u_d = \mathcal{G}v_d. \tag{23}$$

Natural assumptions on $\mathcal{F}$ and $\mathcal{G}$ are given in [**2**], together with an error analysis. In particular, denote an isolated solution of (3) by u^* and let $v^* = \mathcal{F}(u^*)$. Then for d sufficiently large, say $d \geq d_0$, the approximating equation (22) has a solution v_d that is unique in some neighborhood of v^* that is independent of d. Moreover,

$$\|v^* - v_d\| \leq (1+\delta_d)\, \|v^* - \mathcal{P}_d v^*\|, \quad d \geq d_0 \tag{24}$$

with $\delta_d \to 0$.

By the density of the polynomials in $L^2(D)$, we know $\mathcal{P}_d v^* \to v^*$ for all $v^* \in L^2(D)$. Therefore, (24) proves the convergence of $v_d \to v^*$ for all possible cases. Also,

$$u^* - u_d = \mathcal{G}(v^* - v_d)$$
$$\|u^* - u_d\| \leq \|\mathcal{G}\|\, \|v^* - v_d\|.$$

The norm is the standard L^2-norm for $L^2(D)$. This proves the convergence of $u_d \to u^*$ as $d \to \infty$. Additional error analysis results are given in [**2**], much of which are based on [**3**].

3.1 *Speed of convergence.* The key to obtaining results on the rapid convergence of $\{v_s\}$ and $\{u_d\}$ is to look at the speed of convergence for the orthogonal projection operator, $\mathcal{P}_d w \to w$, for $w \in L^2(D)$. For this, we use results from Ragozin [**10**, p. 164] as summarized below.

Assume $w \in C^k(\overline{D})$ with $k \geq 0$ an integer. For the norm on $C^k(\overline{D})$, we use the standard definition

$$\|w\|_{C^k} = \sum_{i+j \leq k} \left\| \frac{\partial^{i+j} w}{\partial x^i \partial y^j} \right\|_\infty .$$

In addition, define various moduli of continuity by

$$\omega(w;h) = \sup\{|w(x_1,y_1) - w(x_2,y_2)| : |(x_1,y_1) - (x_2,y_2)| \leq h\}$$
$$\omega_k(w;h) = \sum_{i+j=k} \omega\left(\frac{\partial^{i+j} w}{\partial x^i \partial y^j}; h\right), \quad k \geq 1$$

Then there exists a sequence of polynomials p_d of degree $\leq d$ such that

$$\|w - p_d\|_\infty \leq \frac{B_k}{d^k}\left[\frac{\|w\|_{C^k}}{d} + \omega_k\left(w; \frac{1}{d}\right)\right], \quad d \geq 1 \tag{25}$$

where each constant B_k depends only on $k \geq 0$.

Apply (25) to $w = v^*$, the solution of (18), and let p_d denote the approximation of v^* that is referenced in (25). Then

$$\begin{aligned} \|v^* - \mathcal{P}_d v^*\| &\leq \|v^* - p_d\| \\ &\leq \sqrt{\pi}\, \|v^* - p_d\|_\infty . \end{aligned}$$

If we assume $v^* \in C^k(\overline{D})$ for some $k \geq 0$, $v^* = f(x,y,u^*(x,y))$, then we can apply (25) to obtain bounds on the rate of convergence of $v_d \to v^*$ within $L^2(D)$.

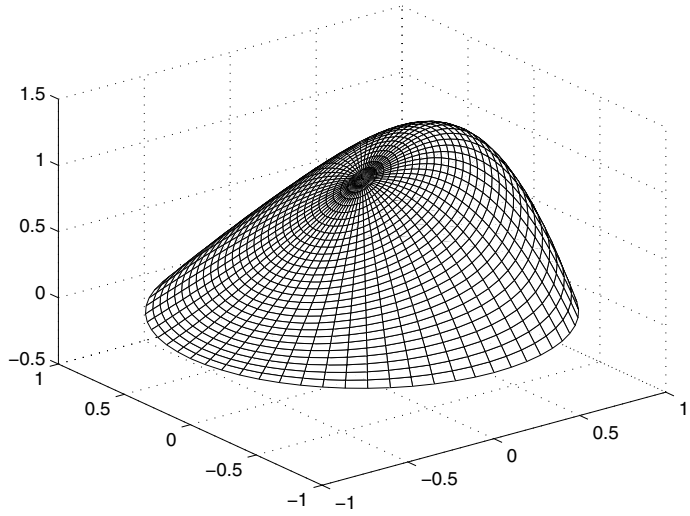

FIGURE 1. The true solution $u = \left(1 - x^2 - y^2\right) e^{x \cos y}$.

4. Numerical example. As a test case to examine the rate of convergence, we solve the problem

$$\begin{aligned} -\Delta u(x,y) &= e^{u(x,y)} + \beta(x,y), \quad x^2 + y^2 \le 1 \\ u(x,y) &= 0, \quad x^2 + y^2 = 1 \end{aligned} \tag{26}$$

with $\beta(x,y)$ chosen such that the true solution is

$$u(x,y) = \left(1 - x^2 - y^2\right) e^{x \cos y}, \qquad x^2 + y^2 \le 1. \tag{27}$$

The true solution is illustrated in Figure 1.

In Table 2 we give numerical results for $d = 1, \ldots, 20$. The error was evaluated using a polar coordinates mesh of approximately 800 points. The linearity of the semi-log graph in Figure 2 illustrates that the convergence is exponential in d.

Newton's method was used to solve the nonlinear system (13). For production code, I would recommend using Broyden's method, cf. [**7**] or a two-grid iterative variant of Newton's method, cf. [**1**]. The integrals in (13) were evaluated numerically using methods from [**11**].

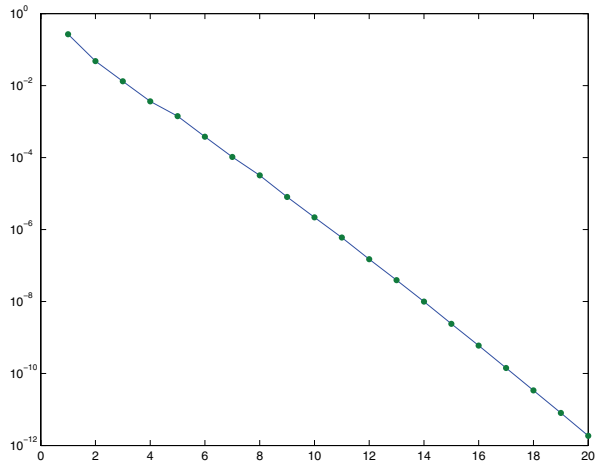

FIGURE 2. Error versus d for the example (26)–(27).

TABLE 2. Maximum errors in 'Fourier solution' u_d.

degree	N	$\|u-u_d\|_\infty$	degree	N	$\|u-u_d\|_\infty$
1	3	2.67E−1	11	78	6.01E−7
2	6	4.79E−2	12	91	1.50E−7
3	10	1.31E−2	13	105	3.92E−8
4	15	3.64E−3	14	120	9.92E−9
5	21	1.41E−3	15	136	2.41E−9
6	28	3.80E−4	16	153	5.99E−10
7	36	1.04E−4	17	171	1.43E−10
8	45	3.20E−5	18	190	3.39E−11
9	55	8.04E−6	19	210	8.05E−12
10	66	2.18E−6	20	231	1.86E−12

5. Construction of an orthogonal basis. In this section we return to the choice of a basis for the Galerkin method described in Section 2. If one looks at the system (13) and the basis $\Phi_{n,k}$ in (15), where

$$\Phi_{n,k} = \Phi_{n,0} \circ R_k^{[n]}, \quad \Phi_{n,0}(x,y) \equiv \frac{1}{\sqrt{\pi}} U_n(x),$$

and $R_k^{[n]}$ is the rotation

$$\begin{aligned} R_k^{[n]} &:= \begin{pmatrix} c_k^{[n]} & s_k^{[n]} \\ -s_k^{[n]} & c_k^{[n]} \end{pmatrix} \\ c_k^{[n]} &:= \cos\left(\frac{k\pi}{n+1}\right), \\ s_k^{[n]} &:= \sin\left(\frac{k\pi}{n+1}\right) \end{aligned}$$

one is tempted to search for polynomials $\hat{\Phi}_{n,k}$ with

$$\begin{aligned} \hat{\Phi}_{n,k} &= \hat{\Phi}_{n,0} \circ R_k^{[n]} \\ \hat{\Lambda}_{n,k} &= -\Delta((1-x^2-y^2)\hat{\Phi}_{n,k}(x,y)) \\ &= \Phi_{n,k}(x,y). \end{aligned} \tag{28}$$

Compared with the basis $\Phi_{n,k}$ the basis $\hat{\Phi}_{n,k}$ has the advantage that the matrix $(\hat{\Lambda}_m, \hat{\Lambda}_n)$ in (13) is the identity matrix and $\hat{\Lambda}_{n,k}$ can be evaluated with the triple recursion for U_n. Formula (28) implies that the polynomials $\hat{\Phi}_{n,k}$ are orthonormal with respect to the nonstandard scalar product

$$\begin{aligned} &(f,g)_{C^2(D)} \\ &\quad := [\Delta((1-x^2-y^2)f(x,y))][\Delta((1-x^2-y^2)g(x,y))]\,dx\,dy \end{aligned} \tag{29}$$

for $f, g \in C^2(D)$. In a previous article, see [**6**], a similar problem in one dimension was studied. From this investigation we learned that it might be an advantage to search for $\hat{\Phi}_{n,0}$ as the pre-image of $\Phi_{n,0}$:

$$\hat{\Phi}_{n,0} \equiv (\Delta \circ M)^{-1}(-\Phi_{n,0}) \tag{30}$$

where M is the multiplication operator

$$(Mf)(x,y) \equiv (1-x^2-y^2)f(x,y).$$

To study the mapping $\Delta \circ M$ we introduce the following subspaces

$$\Pi^e_{2m} := \left\{ \sum_{i=0}^{m} \sum_{k=0}^{i} a_{i-k,k} x^{2(i-k)} y^{2k} \mid a_{j,k} \in \mathbf{R} \right\} \subset \Pi_{2m}$$
$$\Pi^o_{2m+1} := \left\{ \sum_{i=0}^{m} \sum_{k=0}^{i} a_{i-k,k} x^{2(i-k)+1} y^{2k} \mid a_{j,k} \in \mathbf{R} \right\} \subset \Pi_{2m+1}.$$

Then we have the following mapping properties

$$\Pi^e_{2m} \xrightarrow{M} \Pi^e_{2m+2} \xrightarrow{\Delta} \Pi^e_{2m}$$
$$\Pi^o_{2m+1} \xrightarrow{M} \Pi^o_{2m+3} \xrightarrow{\Delta} \Pi^o_{2m+1}.$$

In the following we describe the construction of $\hat{\Phi}_{n,0}$ for even $n = 2m$, and we will only state the result for odd n.

Note that the mapping

$$M : \Pi^e_{2m} \longrightarrow \Pi^e_{2m+2}$$

is one-to-one but not onto. The operator $\Delta : \Pi^e_{2m+2} \to \Pi^e_{2m}$ has the null space

$$\mathcal{N}_{2m+2} \equiv * \operatorname{span} \{k_0(x,y), \ldots, k_{m+1}(x,y)\} \subset \Pi^e_{2m+2}$$
$$k_j(x,y) = \sum_{l=0}^{j} \kappa^j_l x^{2(j-l)} y^{2l} \in \Pi^e_{2j}, \qquad \kappa^j_l = (-1)^l \binom{2j}{2l}.$$

To solve (30) we notice first that the function $\Phi_{n,0}$ does not depend on y and is an element of Π^e_{2m},

$$-\Phi_{n,0}(x,y) = \sum_{j=0}^{m} r_j x^{2j} \equiv -\frac{1}{\sqrt{\pi}} U_n(x).$$

Each polynomial $\overline{q}(x,y)$ of the following form

$$\overline{q}(x,y) := \sum_{j=1}^{m+1} \overline{q}_j x^{2j} + \sum_{j=0}^{m+1} \alpha_j k_j(x,y)$$
$$\overline{q}_j := \frac{r_{j-1}}{(2j)(2j-1)}, \quad j = 1,\dots,m, \quad \alpha_j \in \mathbf{R}$$

will solve the equation

$$\Delta \overline{q} = -\Phi_{n,0}.$$

But we have to choose the coefficients α_j in such a way that we can solve

$$M\hat{\Phi}_{n,0} = \overline{q}. \tag{31}$$

The structure of the multiplication operator M is so simple that we can solve (31) recursively. We decompose the polynomial $\hat{\Phi}_{n,0}$ into its homogeneous components

$$\hat{\Phi}_{n,0}(x,y) = \sum_{i=0}^{m} p_i(x,y), \quad p_i(x,y) = \sum_{k=0}^{i} \overline{p}_{i-k,k} x^{2(i-k)} y^{2k}$$

and solve the following system of equations

$$\begin{aligned}
(M-I)p_m &= \overline{q}_{m+1} x^{2(m+1)} + \alpha_{m+1} k_{m+1} \\
(M-I)p_{m-1} &= -p_m + \overline{q}_m x^{2m} + \alpha_m k_m \\
(M-I)p_{m-2} &= -p_{m-1} + \overline{q}_{m-1} x^{2m-2} + \alpha_{m-1} k_{m-1} \\
&\vdots = \vdots \\
(M-I)p_0 &= -p_1 + \overline{q}_1 x^2 + \alpha_1 k_1
\end{aligned} \tag{32}$$

Now we get

$$\begin{aligned}
(\Delta \circ M)\hat{\Phi}_{n,0} &= \Delta\Big(\sum_{i=0}^{m} M p_i \Big) \\
&= \Delta\Big(p_m + \overline{q}_{m+1} x^{2(m+1)} + \alpha_{m+1} k_{m+1} \\
&\quad + \sum_{i=0}^{m-1} (p_i - p_{i+1} + \overline{q}_{i+1} x^{2(i+1)} + \alpha_{i+1} k_{i+1}) \Big) \\
&= \Delta \overline{q}, \quad \text{we use} \quad \Delta p_0 = 0 \\
&= \Phi_{n,0}.
\end{aligned}$$

To demonstrate that we can solve (32) by back substitution, we choose an arbitrary equation $j < m$

$$(I - M)p_j = p_{j+1} - \overline{q}_{j+1}x^{2(j+1)} - \alpha_{j+1}k_{j+1}$$

(in the case $j = m$ the term p_{j+1} on the righthand side is not present) and rewrite this as an equation for the coefficients of p_j:

$$\begin{pmatrix} 1 & & & & & & \\ 1 & 1 & & & & & \\ & 1 & 1 & & & & \\ & & 1 & \ddots & & & \\ & & & \ddots & \ddots & & \\ & & & & & 1 & \\ & & & & & 1 & 1 \\ & & & & & & 1 \end{pmatrix} \begin{pmatrix} \overline{p}_{j,0} \\ \overline{p}_{j-1,1} \\ \overline{p}_{j-2,2} \\ \vdots \\ \vdots \\ \vdots \\ \overline{p}_{0,j} \end{pmatrix}$$

$$= \begin{pmatrix} \overline{p}_{j+1,0} - \overline{q}_{j+1} \\ \overline{p}_{j,1} \\ \overline{p}_{j-1,2} \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \overline{p}_{0,j+1} \end{pmatrix} - \alpha_{j+1} \begin{pmatrix} \kappa_0^{j+1} \\ \kappa_1^{j+1} \\ \kappa_2^{j+1} \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \kappa_{j+1}^{j+1} \end{pmatrix}$$

where the coefficients $\overline{p}_{j-i,i}$, $i = 0(1)j$, and α_{j+1} are unknown. We rewrite this equation system

$$\begin{pmatrix} 1 & & & & & & & \kappa_0^{j+1} \\ 1 & 1 & & & & & & \kappa_1^{j+1} \\ & 1 & 1 & & & & & \kappa_2^{j+1} \\ & & 1 & \ddots & & & & \vdots \\ & & & \ddots & \ddots & & & \vdots \\ & & & & & 1 & & \vdots \\ & & & & & 1 & 1 & \kappa_l^{j+1} \\ & & & & & & 1 & \kappa_{j+1}^{j+1} \end{pmatrix} \begin{pmatrix} \overline{p}_{j,0} \\ \overline{p}_{j-1,1} \\ \overline{p}_{j-2,2} \\ \vdots \\ \vdots \\ \vdots \\ \overline{p}_{0,j} \\ \alpha_{j+1} \end{pmatrix} = \begin{pmatrix} \overline{p}_{j+1,0} - \overline{q}_{j+1} \\ \overline{p}_{j,1} \\ \overline{p}_{j-1,2} \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \overline{p}_{0,j+1} \end{pmatrix}$$

and eliminate the sub diagonal elements to get

$$\begin{pmatrix} 1 & & & & & & \overline{\kappa}_0^{j+1} \\ & 1 & & & & & \overline{\kappa}_1^{j+1} \\ & & 1 & & & & \overline{\kappa}_2^{j+1} \\ & & & \ddots & & & \vdots \\ & & & & \ddots & & \vdots \\ & & & & & 1 & \vdots \\ & & & & & 1 & \overline{\kappa}_j^{j+1} \\ & & & & & & \overline{\kappa}_{j+1}^{j+1} \end{pmatrix} \begin{pmatrix} \overline{p}_{j,0} \\ \overline{p}_{j-1,1} \\ \overline{p}_{j-2,2} \\ \vdots \\ \vdots \\ \vdots \\ \overline{p}_{0,j} \\ \alpha_{j+1} \end{pmatrix} = \begin{pmatrix} \gamma_0^{j+1} \\ \gamma_1^{j+1} \\ \gamma_2^{j+1} \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \gamma_{j+1}^{j+1} \end{pmatrix}$$

where

$$\begin{aligned} \overline{\kappa}_l^{j+1} &= \sum_{k=0}^{l} (-1)^k \kappa_{l-k}^{j+1} \\ \gamma_l^{j+1} &= \left[\sum_{k=0}^{l} (-1)^{l+k} \overline{p}_{j+1-k,k} \right] + (-1)^{l+1} \overline{q}_{j+1} \end{aligned}$$

for $l = 0, \ldots, j+1$. Again the term in square brackets is omitted in the case $j = m$. We remark that the term $\overline{\kappa}_{j+1}^{j+1}$ is not equal to zero

$$\begin{aligned} \overline{\kappa}_{j+1}^{j+1} &= \sum_{k=0}^{j+1} (-1)^k (-1)^{j+1-k} \binom{2(j+1)}{2(j+1-k)} \\ &= (-1)^{j+1} 2^{2j+1} \neq 0. \end{aligned}$$

This shows that we can calculate the coefficients of $\hat{\Phi}_{n,0}$ with the following algorithm

$$\begin{aligned} \gamma_l^{j+1} &= \left[\sum_{k=0}^{l} (-1)^{l+k} \overline{p}_{j+1-k,k} \right] + (-1)^{l+1} \overline{q}_{j+1}, \quad l = 0, \ldots, j+1 \\ \overline{\kappa}_l^{j+1} &= (-1)^l \sum_{k=0}^{l} \binom{2(j+1)}{2(l-k)}, \quad l = 0, \ldots, j+1 \\ \alpha_{j+1} &= (-1)^{j+1} \frac{\gamma_{j+1}^{j+1}}{2^{2j+1}}, \\ \overline{p}_{j-l,l} &= \gamma_l^{j+1} - \overline{\kappa}_l^{j+1} \alpha_{j+1}, \quad l = 0, \ldots, j, \end{aligned}$$

for $j = m, m-1, \ldots, 0$. The term in square brackets is omitted in the case $j = m$ and the complexity of the above algorithm is $O(m^3)$. The solution $\hat{\Phi}_{n,0}$ of (28) is given by

$$\hat{\Phi}_{n,0}(x,y) = \sum_{i=0}^{m} \sum_{k=0}^{i} \overline{p}_{i-k,k} x^{2(i-k)} y^{2k}.$$

If $n = 2m+1$ is odd we can calculate the coefficients of $\hat{\Phi}_{n,0}$ with the following algorithm

$$\gamma_l^{j+1} = \left[\sum_{k=0}^{l} (-1)^{l+k} \overline{p}_{j+1-k,k} \right] + (-1)^{l+1} \overline{q}_{j+1}, \quad l = 0, \ldots, j+1$$

$$\overline{\kappa}_l^{j+1} = (-1)^l \sum_{k=0}^{l} \binom{2(j+1)+1}{2(l-k)}, \quad l = 0, \ldots, j+1$$

$$\alpha_{j+1} = (-1)^{j+1} \frac{\gamma_{j+1}^{j+1}}{2^{2j+2}},$$

$$\overline{p}_{j-l,l} = \gamma_l^{j+1} - \overline{\kappa}_l^{j+1} \alpha_{j+1}, \quad l = 0, \ldots, j,$$

for $j = m, m-1, \ldots, 0$. Here $\hat{\Phi}_{n,0}$ is given by

$$\hat{\Phi}_{n,0}(x,y) = \sum_{i=0}^{m} \sum_{k=0}^{i} \overline{p}_{i-k,k} x^{2(i-k)+1} y^{2k}.$$

For the odd case the coefficients $\overline{q}_j$ are defined with the help of the coefficients r_j of the Chebyshev polynomials of odd degree:

$$\overline{q}_j = \frac{r_{j-1}}{(2j+1)(2j)}, \quad j = 1, \ldots, m$$

$$-\frac{1}{\sqrt{\pi}}(x) U_n(x) = \sum_{j=0}^{m} r_j x^{2j+1}.$$

An implementation of the Kumar-Sloan algorithm with the above polynomials $\widehat{\Phi}_{n,k}$ shows the same results as in Table 2. But, as we mentioned at the beginning of Section 5, no inversion of the Gram matrix is necessary and the evaluation of $\hat{\Lambda}_{n,k}$ is straightforward.

Concluding remarks. Another approach to constructing polynomials that are orthogonal using the Sobolev-type inner product of (29) is given in [**12**]. It leads to a set of orthogonal polynomials different from those given in Section 5; and his results are given for the unit ball in $\mathbf{R}^d$ with arbitrary $d \geq 2$.

For an illustration of solving the original problem (1) on a region other than the unit disk, namely, an ellipse, see the original paper [**2**]. To solve (2), or (1), with nonzero boundary conditions requires either solving a boundary integral equation for Laplace's equation or the interpolation over $\overline{D}$ of the nonzero boundary condition by some smooth function u_0 followed by a change in the function f and the definition of the unknown solution u being sought.

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