Solving a Single Layer Integral Equation On Surfaces in R^3

Yan Chen and Kendall Atkinson
Department of Mathematics
University of Iowa
Iowa City, IA 52242

January 27, 1994

Abstract

In this paper, we consider solving the single layer integral equation (3) on a closed surface in R^3. The numerical method is based on Galerkin's method with spherical polynomials as the approximationing functions. We study the error of the approximating solution in suitable Sobolev spaces.

Key Words: Spherical polynomials, Sobolev space, Galerkin's method.
AMS (MOS) Subject Classification: $65R20$.

1 Introduction

Consider the solution of the following Dirichlet problem:

$$\begin{cases} \triangle u = 0 & \text{in } \Omega \\ u = u_0 & \text{on } \Gamma \end{cases} \tag{1}$$

with Ω an open bounded simply-connected region in R^3. Assume that $\Gamma = \partial\Omega$, the boundary of Ω, is sufficiently smooth. From [6], the solution can be represented by the expression

$$u(X) = \frac{1}{4\pi}\int_\Gamma \frac{q(Y)}{\mid X - Y \mid} dY \quad , \quad X \in \overline{\Omega} \tag{2}$$

in which $q(Y)$ is called a single-layer density function and is determined by solving

$$\frac{1}{4\pi}\int_{\Gamma}\frac{q(Y)}{\mid X-Y\mid}dY = u_0 \quad , \quad X\in\Gamma \tag{3}$$

In this paper, we solve (3) by Galerkin's method with approximations based on spherical harmonics on the unit sphere. We study the error of the solution in suitable Sobolev spaces, and we give both a practical implementation of the numerical method and numerical examples. The details are presented below. Briefly, the numerical method is presented in section 2, and the practical implementation of the numerical method is covered in section 3. In section 4, we give numerical examples.

2 The Numerical Method

Let Sobolev spaces $H^t(G)$ ($G\subseteq\mathbf{R}^m$ with G having a non-empty interior, $t\in\mathbf{R}$) be defined as follows: For an integer $t\geq 0$, let

$$H^t(G) = \{f \mid \|f\|_t^2 = \sum_{|j|\leq t}\|\partial^j f\|^2_{L^2(G)} < \infty\}$$

The derivatives are distributional derivatives. For $t>0$ and not an integer, let $t=p+b$, p an integer, and $0<b<1$. Define

$$H^t(G) = \{f \mid \|f\|_t^2 = \|f\|_p^2 + \sum_{|j|=p}\int_G\int_G\frac{|\partial^j f(x)-\partial^j f(y)|^2}{|x-y|^{m+2b}}d\sigma(x)d\sigma(y) < \infty\} \tag{4}$$

for $j=(j_1,\ldots,j_m)$, $\partial^j f(x) = \frac{\partial^{|j|}f(x_1,\ldots,x_m)}{\partial x_1^{j_1}\cdots\partial x_m^{j_m}}$. For $t<0$, let

$$H^t = (H^{-t})^*$$

the dual space of $H^{-t}(G)$; see [9].

We suppose Γ is regular (see [6]). Then we assume that

$$\Gamma = \bigcup_{i=1}^{k}\Gamma_i \quad \text{and} \quad \Gamma_i = \{(x,y,F_i(x,y)) \mid (x,y)\in\Lambda_i\} \tag{5}$$

where Λ_i is an open bounded simply-connected region in R^2, and F_i is a smooth function on Λ_i, $i = 1, \cdots, k$. For any function g on Γ, define

$$\tilde{g}_i(x, y) = g_i(x, y, F_i(x, y)), \qquad g_i = g|\Gamma_i, \quad i = 1, \cdots, k$$

The Sobolev spaces $H^t(\Gamma)$ for any real number t are defined as follows:

$$H^t(\Gamma) \equiv \{g \mid \|g\|_{H^t(\Gamma)} = \sqrt{\sum_{i=1}^{k} \|\tilde{g}_i\|^2_{H^t(\Lambda_i)}} < \infty\}, \qquad x \in \Gamma$$

Let

$$\mathcal{K}q(X) = \frac{1}{4\pi} \int_\Gamma \frac{q(Y)}{\mid X - Y \mid} dY$$

Then (3) can be written as

$$\mathcal{K}q(X) = u_0(X), \qquad X \in \Gamma \tag{6}$$

From [6], $\mathcal{K}$ is an isomorphism of the Hilbert space $H^{-\frac{1}{2}}(\Gamma)$ onto the Hilbert space $H^{\frac{1}{2}}(\Gamma)$; and the associated bilinear form

$$a(q, q') = \frac{1}{4\pi} \int_\Gamma \int_\Gamma \frac{q(X) q'(Y)}{\mid X - Y \mid} dX dY$$

satisfies a strong ellipticity condition:

$$a(q, q') \geq C\|q\|^2_{H^{-\frac{1}{2}}(\Gamma)}, \qquad q \in H^{-\frac{1}{2}}(\Gamma), \quad C > 0. \tag{7}$$

Then it can be shown that solving (6) converts to the problem of finding $q \in H^{-\frac{1}{2}}(\Gamma)$ for which

$$a(q, q') = (u_0, q'), \quad \text{all} \quad q' \in H^{-\frac{1}{2}}(\Gamma) \tag{8}$$

2.1 Construction of a finite dimensional subspace of $H^{-\frac{1}{2}}(\Gamma)$

Let U denote the unit sphere in R^3,

$$U = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$$

From [7], there is a standard orthogonal basis of spherical harmonics for $L^2(U)$. Let $P_n(u)$ and $P_n^m(u)$ be the Legendre polynomials and associated Legendre functions on $[-1,1]$, $n \geq 0$, $1 \leq m \leq n$. For $(x,y,z) \in U$, let

$$(x,y,z) = (\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta)$$

for $0 \leq \phi \leq 2\pi$, $0 \leq \theta \leq \pi$. The basis functions for the spherical harmonics of degree n are

$$P_n(\cos\theta), P_n^m(\cos\theta\cos m\phi), P_n^m(\cos\theta\sin m\phi), \quad 1 \leq m \leq n$$

The total number of basis functions of degree $\leq N$ is $d_N = (N+1)^2$. Let $H_1^0 \cdots H_{d_N}^0$ be normalized basis functions in $L^2(\Gamma)$ and let $\mathcal{X}_N$ be the space of spherical polynomials of degree $\leq N$:

$$\mathcal{X}_N = \text{Span}\{H_1^0 \cdots H_{d_N}^0\} = \{\sum_{i=1}^{d_N} a_i H_i^0 \mid a_i \in R\}$$

Then easily $\mathcal{X}_N \subseteq H^{-\frac{1}{2}}(U)$.

Assume that there is a C^∞ mapping $\bar{M} : U \xrightarrow[\text{onto}]{1\text{–}1} \Gamma$. For any function g defined on Γ, we define $\hat{g}$ on U, by

$$\hat{g}(\hat{Q}) = g(\bar{M}(\hat{Q}))$$

for any $\hat{Q} \in U$. This defines a map $\mathcal{M} : L^2(\Gamma) \to L^2(U)$, with $\mathcal{M}g = \hat{g}$. Let V_N be the finite dimensional space:

$$V_N = \text{Span}\{\eta_1 \cdots \eta_{d_N}\}$$

where

$$\eta_i(Q) = H_i^0(\bar{M}^{-1}(Q))$$

for $Q \in \Gamma$, $i = 1, \cdots, d_N$. The result that V_N is a subspace of $H^{-\frac{1}{2}}(\Gamma)$ or that V_N is a subspace of all Sobolev spaces $H^t(\Gamma)$ $(t \in R)$, is obtained immediately from the following theorem.

Theorem 2.2 *Assume $\bar{M} : U \xrightarrow[\text{onto}]{1\text{–}1} \Gamma$ is a C^∞ mapping. Then for any $t \geq 0$, the associated linear mapping $\mathcal{M} : H^t(\Gamma) \xrightarrow[\text{onto}]{1\text{–}1} H^t(U)$ and $\|\mathcal{M}f\|_{H^t(U)}$ is equivalent to $\|f\|_{H^t(\Gamma)}$, for any $f \in H^t(\Gamma)$. By duality, the map $\mathcal{M}$ can be extended uniquely to a bounded map of $H^{-t}(\Gamma) \xrightarrow[\text{onto}]{1\text{–}1} H^{-t}(U)$.*

In order to prove the theorem, we need some lemmas.

Lemma 2.3 *Let $r \in \mathbf{R}$, Then $g \in H^r(U)$ if and only if its Laplace expansion satisfies*

$$\sum_{n=0}^{\infty}(2n+1)^{2r}\{\mid a_n \mid^2 + \sum_{m=1}^{n}[\mid a_n^m \mid^2 + \mid b_n^m \mid^2]\} < \infty$$

where

$$g = \sum_{n=0}^{\infty}\{a_n H_n + \sum_{m=1}^{n}[a_n^m H_{n1}^m + b_n^m H_{n2}^m]\}$$

In this, H_n, H_{n1}^m, H_{n2}^m, $1 \leq m \leq n$, are orthonormal spherical harmonic basis functions, discussed earlier.

Proof: See [11], Theorem 6.5 on p.264. □

Lemma 2.4 *Let $g \in H^r(U)$, $r \in \mathbf{R}$, and define*

$$\|g\|_{*H^r(U)} = (\sum_{n=0}^{\infty}(2n+1)^{2r}\{|a_n|^2 + \sum_{m=1}^{n}[|a_n^m|^2 + |b_n^m|^2]\})^{\frac{1}{2}}$$

where

$$g = \sum_{n=0}^{\infty}\{a_n H_n + \sum_{m=1}^{n}[a_n^m H_{n1}^m + b_n^m H_{n2}^m]\}$$

Then $\| \cdot \|_{*H^r}$ is equivalent to the usual H^r norm $\| \cdot \|_{H^r}$.

Proof: See [11, p. 215 and p. 259] □

Lemma 2.5 *Define*

$$\langle g_1, g_2 \rangle_{*L^2(\Gamma)} = \int_\Gamma W(Q) g_1(Q) g_2(Q) ds$$

where

$$W(\bar{M}(\hat{Q})) = \frac{1}{J_{\bar{M}}(\hat{Q})}$$

for $\hat{Q} \in U$ with $J_{\bar{M}}$ the Jacobian of the mapping $\bar{M}$. Then $\| \cdot \|_{*L^2(\Gamma)}$ is equivalent to $\| \cdot \|_{L^2(\Gamma)}$ and $\|g\|_{*L^2(\Gamma)} = \|\hat{g}\|_{L^2(U)}$ with

$$\hat{g}(\hat{Q}) = g(\bar{M}(\hat{Q})), \quad \hat{Q} \in U.$$

Proof: Since $\bar{M}$ is a one-to-one C^∞ mapping of U onto Γ, we can assume

$$0 < D_1 \leq J_{\bar{M}}(\hat{Q}) \leq D_2 < \infty$$

for some constants D_1, D_2. Hence

$$0 < C_1 \leq W(Q) \leq C_2 < \infty$$

for suitable constants C_1, C_2 and $Q \in \Gamma$. Then

$$\begin{aligned} C_1 \|g\|_{L^2(\Gamma)} &= C_1 \int_\Gamma |g|^2 ds \\ &\leq \|g\|_{*L^2(\Gamma)} = \int_\Gamma |W(Q)||g|^2 ds \\ &\leq C_2 \int_\Gamma |g|^2 = C_2 \|g\|_{L^2(\Gamma)} \end{aligned}$$

Thus $\|\cdot\|_{*L^2(\Gamma)}$ is equivalent to $\|\cdot\|_{L^2(\Gamma)}$, and

$$\begin{aligned} \langle g_1, g_2 \rangle_{*L^2(\Gamma)} &= \int_\Gamma W(Q) g_1(Q) g_2(Q) ds \\ &= \int_U W(\bar{M}(\hat{Q}_1)) g_1(\bar{M}(\hat{Q}_1)) g_2(\bar{M}(\hat{Q}_1)) J_{\bar{M}}(\hat{Q}) ds \\ &= \int_U \hat{g}_1(\hat{Q}) \hat{g}_2(\hat{Q}) ds \\ &= \langle \hat{g}_1, \hat{g}_2 \rangle_{L^2(U)} \end{aligned}$$

Hence

$$\|g\|_{*L^2(\Gamma)} = \|\hat{g}\|_{L^2(U)}$$

□

Proof Theorem 2.2: We have assumed Γ to be regular as mentioned. Now we only prove this theorem for the case of one of the sub-surfaces Γ_i of (5),

$$\Gamma_i = \{(x, y, F_i(x, y)) \mid (x, y) \in \Lambda_i\}$$

with F_i infinitely differentiable and Λ_i an open region in R^2. For any $g \in H^t(\Gamma)$, define

$$\tilde{g}(x, y) = g(x, y, F_i(x, y))$$

and

$$\|g\|_{H^t(\Gamma_i)} = \sqrt{\|\tilde{g}\|^2_{H^t(\Lambda_i)}}$$

Recall that $\hat{g}(Q) = g(\bar{M}(Q))$ where

$$\bar{M} : U \to \Gamma$$

is a 1-1 and onto smooth map, we can assume that

$$U_i = \bar{M}^{-1}(\Gamma_i)$$

and

$$U_i = \{(\xi, \eta, G_i(\xi, \eta)) | (\xi, \eta) \in \Delta_i\}$$

where Δ_i is an open region in R^2, G_i is infinitely differentiable and

$$\hat{\hat{g}}(\xi, \eta) = \hat{g}(\xi, \eta, G_i(\xi, \eta))$$

Hence

$$\|\hat{g}\|_{H^t(U_i)} = \sqrt{\|\hat{\hat{g}}\|^2_{H^t(\Delta_i)}} \quad \text{if } \hat{g} \in H^t(U_i).$$

Define a projection map P on Λ_i by

$$P : \Gamma_i \to \Lambda_i, \quad \text{by } P(x, y, F_i(x, y)) = (x, y);$$

a map $L : \Delta_i \to \Lambda_i$ by

$$L(\xi, \eta) = P\bar{M}(\xi, \eta, G_i(\xi, \eta))$$

and a map $T : \Lambda_i \to R^2$ by

$$T(x, y) = (\frac{\partial L^{-1}(x, y)}{\partial x}, \frac{\partial L^{-1}(x, y)}{\partial y})$$

Since $G_i, P, \bar{M}$ are all smooth maps, so are L and T.
Case 1. $t = 1$,

$$\begin{aligned} \|g\|^2_{H^1(\Gamma_i)} &= \int_{\Lambda_i} | \tilde{g}|^2 ds_{\Lambda_i} + \int_{\Lambda_i} |grad(\tilde{g})|^2 ds_{\Lambda_i} \\ &= \int_{\Delta_i} |\tilde{g}(L(\xi, \eta))|^2 |J_L| ds_{\Delta_i} + \int_{\Delta_i} |grad(\tilde{g}(L(\xi, \eta))) \begin{pmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \end{pmatrix} |^2 |J_L| ds_{\Delta_i} \end{aligned}$$

where J_L is the Jacobian matrix for the mapping L. By the smoothness of L and L^{-1}, we can suppose that

$$|J_L| \leq C_2 < \infty$$

$$\left|\begin{pmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \end{pmatrix}\right| \leq C_3 < \infty$$

for C_2, C_3 are constants, so

$$\begin{aligned}
\|g\|^2_{H^1(\Gamma_i)} &\leq \int_{\Delta_i} C_2|\hat{\tilde{g}}|^2 ds_{\Delta_i} + \int_{\Delta_i} |grad(\hat{\tilde{g}})|^2 C_3^2 C_2 ds_{\Delta_i} \\
&\leq C(\int_{\Delta_i} |\hat{\tilde{g}}|^2 ds_{\Delta_i} + \int_{\Delta_i} |grad(\hat{\tilde{g}})|^2 ds_{\Delta_i}) \\
&= C\|\hat{g}\|^2_{H^1(U_i)}
\end{aligned}$$

for some constant $C < \infty$.

Similarly, we have

$$\|\hat{g}\|^2_{H^1(U_i)} \leq C\|g\|^2_{H^1(\Gamma_i)}$$

for C some constant.

Case 2. $t = 1 + b, \quad 0 < b < 1$. Recall that T, L, L^{-1} are all smooth, for any $(\xi_1, \eta_1), (\xi_2, \eta_2) \in \Lambda_i$, we have the following:

$$|T(L(\xi_1, \eta_1)) - T(L(\xi_2, \eta_2))| \leq C|(\xi_1, \eta_1) - (\xi_2, \eta_2)|$$

$$|L(\xi_1, \eta_1) - L(\xi_2, \eta_2)| \geq C|(\xi_1, \eta_1) - (\xi_2, \eta_2)|$$

and

$$\max_{(\xi,\eta)\in\Lambda_i} |T(L(\xi, \eta))| \leq C$$

with C a generic constant.

$$\begin{aligned}
\|g\|^2_{H^{1+b}(\Gamma_i)} &= \|\tilde{g}\|^2_{H^1(\Gamma_i)} + \int_{\Lambda_i}\int_{\Lambda_i} \frac{|\partial\tilde{g}(x_1, y_1) - \partial\tilde{g}(x_2, y_2)|^2}{|(x_1, y_1) - (x_2, y_2)|^{2+2b}} dS \\
&\leq C\|\hat{\tilde{g}}\|^2_{H^1(U_i)} + \int_{\Delta_i}\int_{\Delta_i} \frac{\partial\tilde{g}(L(\xi_1, \eta_1)) - \partial\tilde{g}(L(\xi_2, \eta_2))|^2}{|L(\xi_1, \eta_1)) - L(\xi_2, \eta_2)|^{2+2b}} |J_L| dS \\
&\leq C\|\hat{\tilde{g}}\|^2_{H^1(U_i)} \\
&+ C\int_{\Delta_i}\int_{\Delta_i} \frac{|\partial\hat{\tilde{g}}(\xi_1, \eta_1) - \partial\hat{\tilde{g}}(\xi_2, \eta_2)|^2}{|(\xi_1, \eta_1) - (\xi_2, \eta_2)|^{2+2b}} \frac{|(\xi_1, \eta_1) - (\xi_2, \eta_2)|^{2+2b}}{|L(\xi_1, \eta_1) - L(\xi_2, \eta_2)|^{2+2b}} |T(L(\xi_1, \eta_1))||J_L| dS \\
&+ \int_{\Delta_i}\int_{\Delta_i} |\partial\hat{\tilde{g}}(\xi_2, \eta_2)|^2 \frac{|T(L(\xi_1, \eta_1)) - T(L(\xi_2, \eta_2))|^2}{|L(\xi_1, \eta_1) - L(\xi_2, \eta_2)|^{2+2b}} |J_L| dS
\end{aligned}$$

$$
\begin{aligned}
&\le C\|\hat{\tilde{g}}\|^2_{H^1(U_i)} + C\int_{\Delta_i}\int_{\Delta_i} \frac{|\partial\hat{\tilde{g}}(\xi_1,\eta_1) - \partial\hat{\tilde{g}}(\xi_2,\eta_2)|^2}{|(\xi_1,\eta_1)-(\xi_2,\eta_2)|^{2+2b}} dS \\
&+ C\int_{\Delta_i}\int_{\Delta_i} |\partial\hat{\tilde{g}}(\xi_2,\eta_2)|^2 \frac{1}{|(\xi_1,\eta_1)-(\xi_2,\eta_2)|^{2b}} dS \\
&\le C\|\hat{\tilde{g}}\|^2_{H^{1+b}(U_i)} + C\int_{\Delta_i} |\partial\hat{\tilde{g}}|^2 dS \int_{\Delta_i} \frac{1}{|(\xi_1,\eta_1)-(\xi_2,\eta_2)|^{2b}} d\xi_1 d\eta_1 \\
&\le C\|\hat{\tilde{g}}\|^2_{H^{1+b}(U_i)} + C\|\hat{\tilde{g}}\|^2_{H^1(U_i)} \int_{\Delta_i} \frac{1}{|(\xi_1,\eta_1)-(\xi_2,\eta_2)|^{2b}} d\xi_1 d\eta_1 \\
&\le C\|\hat{g}\|^2_{H^{1+b}(U_i)}
\end{aligned}
$$

since

$$\int_{\Delta_i} \frac{1}{|(\xi_1,\eta_1)-(\xi_2,\eta_2)|^{2b}} d\xi_1 d\eta_1 \le C < \infty$$

for $0 < b < 1$.

Similarly,

$$\|\hat{g}\|^2_{H^{1+b}(U_i)} \le C\|g\|^2_{H^{1+b}(\Gamma_i)}$$

Case 3. $t > 0$, an integer. This is similar to **Case 1**, and we omit it.
Case 4. $t > 0$, non-integer. Let $t = m + b$ with m an integer and $0 < b < 1$. This is similar to **Case 2**, and we omit it.
Case 5. $t < 0, t \in \mathbf{R}$. Then we treat $H^t(\Gamma)$ as the dual space of $H^{-t}(\Gamma)$. Let $t = -\tau$ with $\tau \ge 0$, then

$$H^{-\tau}(\Gamma) = (H^\tau(\Gamma))'$$

$$H^{-\tau}(U) = (H^\tau(U))'$$

For any $f \in H^{-\tau}(\Gamma)$,

$$\|f\| = \sup_{v\in H^\tau(\Gamma)} \frac{f(v)}{\|v\|_{H^\tau(\Gamma)}}$$

is equivalent to

$$\|\hat{f}\| = \sup_{u\in H^\tau(U)} \frac{\hat{f}(u)}{\|u\|_{H^\tau(U)}}$$

with $\hat{f} \in H^{-\tau}(U)$ defined by

$$\hat{f}(u) = f(v)$$

where $\hat{v} = u$ for any $u \in H^{\tau}(U)$.

Now we define :

$$\mathcal{M} : H^t(\Gamma) \to H^t(U)$$

by

$$\mathcal{M}f = \hat{f}, \quad \text{for any} \quad f \in H^t(\Gamma)$$

Then $\mathcal{M}$ is 1-1 and onto linear map, and $\|\hat{f}\|_{H^t(U)}$ is equivalent to $\|f\|_{H^t(\Gamma)}$. □

2.2 The Galerkin method and error analysis

We approximate the solution q of (8) on the finite dimensional subspace

$$V_N = \text{Span}\{\eta_1, \cdots \eta_{d_N}\}$$

where

$$\eta_i = H_i^0 \bar{M}^{-1} = \mathcal{M}^{-1} H_i^0$$

from Section 2.1. Let $q_N = \sum_{j=1}^{d_N} \alpha_j \eta_j \in V_N$ be the solution of the Galerkin equation

$$a(q_N, q') = (u_0, q'), \quad \text{all} \quad q' \in V_N$$

In particular,

$$a(q_N, \eta_i) = (u_0, \eta_i), \quad i = 1, 2, \cdots d_N.$$

Hence our Galerkin's method for solving (8) is given by

$$\begin{cases} q_N = \sum_{j=1}^{d_N} \alpha_j \eta_j \\ \\ \sum_{j=1}^{d_N} \alpha_j a_j(\eta_j, \eta_i) = (u_0, \eta_i), \quad i = 1, 2, \cdots, d_N \end{cases} \tag{9}$$

Theorem 2.6 *Let q, q_N be defined on Γ as above, and let $q \in L^2(\Gamma)$. Then q_N exists, is unique, and*

$$\|q - q_N\|_{H^{-\frac{1}{2}}(\Gamma)} \to 0, \quad (N \to \infty)$$

This says the method (9) is convergent.

Proof: q_N exists by using the strong ellipticity condition (7),

$$a(q,q) \geq C\|q\|^2_{H^{-\frac{1}{2}}(\Gamma)}, \quad q \in H^{-\frac{1}{2}}(\Gamma)$$

and the Lax-Milgram theorem; see [10, p. 171]. From this reference,

$$\|q - q_N\|_{H^{-\frac{1}{2}}(\Gamma)} \leq C \inf_{p \in V_N} \|q - p\|_{H^{-\frac{1}{2}}(\Gamma)} \leq C \inf_{p \in V_N} \|q - p\|_{*L^2(\Gamma)} \tag{10}$$

with C a constant independent of N. Note that

$$\|q - p\|_{*L^2(\Gamma)} = \|\hat{q} - \hat{p}\|_{L^2(U)}$$

with

$$\hat{p} = \sum_1^{d_N} \alpha_i H_i^0$$

Thus

$$\inf_{p \in V_N} \|q - p\|_{*L^2(\Gamma)} = \inf_{p \in \mathcal{X}_N} \|\hat{q} - \hat{p}\|_{L^2(U)} = \|\hat{q} - \mathcal{P}_N \hat{q}\|_{L^2(U)}$$

Returning to (10)

$$\|q - q_N\|_{H^{-\frac{1}{2}}(\Gamma)} \leq C\|\hat{q} - \mathcal{P}_N \hat{q}\|_{L^2(U)} \to 0.$$

□

In order to get a more useful error bound, we need the following two properties.

Lemma 2.7 *(Approximation property) Let $s, t \in \mathbf{R}$ with $t \leq s$, Then for any $u \in H^s(U)$, there is an element $u_N \in \mathcal{X}_N$ with*

$$\|u - u_N\|_{H^t(U)} \leq CN^{t-s}\|u\|_{H^s(U)}$$

namely $u_N = \mathcal{P}_N u$.

Proof: By Lemma 2.3 and Lemma 2.4, for $r \in \mathbf{R}$,

$$H^r(U) = \{g \mid \|g\|_{*H^r(U)} < \infty\}$$

where

$$g = \sum_{n=0}^{\infty} \{a_n H_n + \sum_{m=1}^{n} [a_n^m H_{n1}^m + b_n^m H_{n2}^m]\}$$

and

$$\|g\|_{*H^r(U)} = \left[\sum_{n=0}^{\infty}(2n+1)^{2r}\{| a_n |^2 + \sum_{m=1}^{n}[| a_n^m |^2 + | b_n^m |^2]\}\right]^{\frac{1}{2}}$$

is equivalent to $\|g\|_{H^r(U)}$. Now for $u \in H^s(U)$, suppose

$$u = \sum_{n=0}^{\infty}\{a_n H_n + \sum_{m=1}^{n}[a_n^m H_{n1}^m + b_n^m H_{n2}^m]\}$$

Let

$$u_N = \sum_{n=0}^{N}\{a_n H_n + \sum_{m=1}^{n}[a_n^m H_{n1}^m + b_n^m H_{n2}^m]\}$$

Hence $u_N \in \mathcal{X}_N$, and

$$\begin{aligned}
&\|u - u_N\|_{H^t(U)}^2 \leq C\|u - u_N\|_{*H^t(U)}^2 \\
= \;& C \sum_{n=N}^{\infty}(2n+1)^{2t}\{| a_n |^2 + \sum_{m=1}^{n}[| a_n^m |^2 + | b_n^m |^2]\} \\
= \;& C \sum_{n=N}^{\infty}(2n+1)^{2t-2s}(2n+1)^{2s}\{| a_n |^2 + \sum_{m=1}^{n}[| a_n^m |^2 + | b_n^m |^2]\} \\
\leq \;& C(2N)^{2(t-s)}\sum_{n=0}^{\infty}(2n+1)^{2s}\{| a_n |^2 + \sum_{m=1}^{n}[| a_n^m |^2 + | b_n^m |^2]\} \\
\leq \;& CN^{2(t-s)}\|u\|_{*H^s(U)}^2 \\
\leq \;& CN^{2(t-s)}\|u\|_{H^s(U)}^2 \quad \text{for some constant C}
\end{aligned}$$

So

$$\|u - u_N\|_{H^t(U)} \leq CN^{t-s}\|u\|_{H^s(U)}$$

□

Lemma 2.8 *(Inverse property) Let $t, s \in \mathbf{R}$ with $t \leq s$. Then for any $v \in \mathcal{X}_{\mathcal{N}}$,*

$$\|v\|_{H^s(U)} \leq CN^{s-t}\|v\|_{H^t(U)}$$

with C independent of v and N.

Proof: For any $v \in \mathcal{X}_{\mathcal{N}}$,

$$v = \sum_{n=0}^{N} \{a_n H_n + \sum_{m=1}^{n} [a_n^m H_{n1}^m + b_n^m H_{n2}^m]\}$$

and

$$\begin{aligned}
\|v\|_{H^s(U)}^2 &\leq C\|v\|_{*H^s(U)}^2 \\
&= C\sum_{n=0}^{N}(2n+1)^{2s}\{| a_n |^2 + \sum_{m=1}^{n}[| a_n^m |^2 + | b_n^m |^2]\} \\
&\leq C\sum_{n=0}^{N}(2n+1)^{2s-2t}(2n+1)^{2t}\{| a_n |^2 + \sum_{m=1}^{n}[| a_n^m |^2 + | b_n^m |^2]\} \\
&\leq C(2N+1)^{2(s-t)}\sum_{n=0}^{N}(2n+1)^{2t}\{| a_n |^2 + \sum_{m=1}^{n}[| a_n^m |^2 + | b_n^m |^2]\} \\
&\leq CN^{2(s-t)}\|v\|_{*H^t(U)}^2 \leq CN^{2(s-t)}\|v\|_{H^t(U)}^2
\end{aligned}$$

So

$$\|v\|_{H^s(U)} \leq CN^{s-t}\|v\|_{H^t(U)}$$

□

Theorem 2.9 *Let q and q_N be defined as above in (6) and (9). If $q \in H^s(\Gamma)$, any $s \in \mathbf{R}, s > -\frac{1}{2}$, then for any $-\frac{1}{2} < r \leq s$,*

$$\|q - q_N\|_{H^r(\Gamma)} = O(\frac{1}{N^{s-r}})$$

Proof: Define a projection

$$\mathcal{L}_N : H^{-\frac{1}{2}}(\Gamma) \to V_N \subset H^{-\frac{1}{2}}(\Gamma)$$

such that

$$\mathcal{L}_N q = q_N \quad \text{for any } q \in H^{-\frac{1}{2}}(\Gamma)$$

Then $\mathcal{L}_N$ is a bounded projection. In fact, by the strong ellipticity condition (7),

$$a(q,q) \geq \alpha\|q\|_{H^{-\frac{1}{2}}(\Gamma)}^2, \quad \alpha > 0$$

This means that for any $q \in H^{-\frac{1}{2}}(\Gamma)$ and

$$\mathcal{K}q(X) = \frac{1}{4\pi}\int_\Gamma \frac{q(Y)}{|X-Y|}$$

we get

$$(\mathcal{K}q, q) \geq \alpha \|q\|^2_{H^{-\frac{1}{2}}(\Gamma)}$$

From [14, p. 9],

$$\|\mathcal{L}_N q\|_{H^{-\frac{1}{2}}(\Gamma)} = \|q_N\|_{H^{-\frac{1}{2}}(\Gamma)} \leq \frac{\|\mathcal{K}\|}{\alpha}\|q\|^2_{H^{-\frac{1}{2}}(\Gamma)}$$

That is

$$\|\mathcal{L}_N\| \leq \frac{\|\mathcal{K}\|}{\alpha} < \infty$$

and $\mathcal{L}_N$ is bounded.

If $q \in V_N$ then the solution of

$$a(q, q^{'}) = (u_0, q^{'}) \quad , \quad \text{all } q^{'} \in V_N$$

is q itself. Hence

$$q_N = q \quad \text{or} \quad \mathcal{L}_N q = q, \quad \text{any} \quad q \in V_N.$$

Thus $\mathcal{L}_N$ is a projection.

Now for $q \in H^r(\Gamma)$, $r \in \mathbf{R}$, by Theorem 2.2, we have a unique $\hat{q} \in H^r(U)$ with $\mathcal{M}q = \hat{q}$. By the Approximation Property, for any $s \geq r$, we have $\hat{q}_{1,N} \in \mathcal{X}_\mathcal{N}$, $(q_{1,N} \in V_N)$ for the orthogonal projection that

$$\|\hat{q} - \hat{q}_{1,N}\|_{H^r(U)} \leq C N^{r-s} \|\hat{q}\|_{H^s(U)}$$

Also

$$\begin{aligned}
\|q - q_N\|_{H^r(\Gamma)} &\leq C\|\hat{q} - \hat{q}_N\|_{H^r(U)} \\
&\leq C\left[\|\hat{q} - \hat{q}_{1,N}\|_{H^r(U)} + \|\hat{q}_{1,N} - \hat{q}_N\|_{H^r(U)}\right] \\
&\leq C\left[\|q - q_{1,N}\|_{H^r(\Gamma)} + \|q_{1,N} - q_N\|_{H^r(\Gamma)}\right] \\
&\leq C\left[\|q - q_{1,N}\|_{H^r(\Gamma)} + \|\mathcal{L}_N(q_{1,N} - q)\|_{H^r(\Gamma)}\right]
\end{aligned}$$

From the Inverse Property,

$$\|\hat{q}_{1,N} - \hat{q}_N\|_{H^r(U)} \leq CN^{r+\frac{1}{2}} \|\hat{q}_{1,N} - \hat{q}_N\|_{H^{-\frac{1}{2}}(U)}$$

Then we have

$$\|\mathcal{L}_N(q_{1,N} - q)\|_{H^r(\Gamma)} \leq CN^{r+\frac{1}{2}} \|\mathcal{L}_N(q_{1,N} - q)\|_{H^{-\frac{1}{2}}(\Gamma)}$$

Hence

$$\begin{aligned}
& \|q - q_{1,N}\|_{H^r(\Gamma)} + \|\mathcal{L}_N(q_{1,N} - q)\|_{H^r(\Gamma)} \\
\leq\ & C\|q - q_{1,N}\|_{H^r(\Gamma)} + CN^{r+\frac{1}{2}} \|\mathcal{L}_N(q_{1,N} - q)\|_{H^{-\frac{1}{2}}(\Gamma)} \\
\leq\ & C\|q - q_{1,N}\|_{H^r(\Gamma)} + CN^{r+\frac{1}{2}} \|q_{1,N} - q\|_{H^{-\frac{1}{2}}(\Gamma)} \\
\leq\ & C\|q - q_{1,N}\|_{H^r(\Gamma)} + CN^{r+\frac{1}{2}} N^{-\frac{1}{2}-r} \|q_{1,N} - q\|_{H^r(\Gamma)} \\
\leq\ & C\|q_{1,N} - q\|_{H^r(\Gamma)} \\
\leq\ & C\|\hat{q}_{1,N} - \hat{q}\|_{H^r(U)} \\
\leq\ & CN^{r-s} \|\hat{q}\|_{H^s(U)} \\
\leq\ & CN^{r-s} \|q\|_{H^s(\Gamma)}
\end{aligned}$$

So

$$\|q - q_N\|_{H^r(\Gamma)} = O(\frac{1}{N^{s-r}})$$

for any $s \geq r$, and $q \in H^s(\Gamma)$.

□

2.3 Condition number of system

We approximate the solution q of (8) by Galerkin's method on V_N. The method leads to solving the usually small linear system

$$B\alpha = b$$

where

$$B = (a_{i,j})_{d_N \times d_N}, \qquad a_{i,j} = a(\eta_i, \eta_j) \quad i, j = 1, 2, \cdots, d_N$$

$$\alpha^T = (\alpha_1, \cdots, \alpha_{d_N}), \qquad b^T = ((u_0, \eta_1), \cdots, (u_0, \eta_{d_N}))$$

The following theorem gives an upper bound for the condition number of the matrix B.

Theorem 2.10 *The condition number of the matrix B is $O(N)$:*

$$Cond(B) = \|B\|\|B^{-1}\| \leq CN$$

for C a constant.

Proof We define the matrix norm by

$$\|B\| = \max_{\eta \in R^M, \eta \neq 0} \frac{|B\eta|}{|\eta|}$$

It is known that the smallest and largest eigenvalues of the symmetric positive definite matrix B satisfy

$$0 < \lambda_{min} = \min_{\eta \in R^M, \eta \neq 0} \frac{\eta B \eta}{|\eta|^2}, \qquad \lambda_{max} = \max_{\eta \in R^M, \eta \neq 0} \frac{\eta B \eta}{|\eta|^2}$$

where $|\cdot|$ denotes the usual Euclidean norm,

$$|\eta| = (\sum_{i=1}^{M} \eta_i^2)^{\frac{1}{2}}$$

Thus the condition number of B is given by

$$Cond(B) = \|B\|\|B^{-1}\| = \frac{\lambda_{max}}{\lambda_{min}}$$

See [12, pp. 126-128].

Recall that

$$(\eta_i, \eta_j)_{*L^2(\Gamma)} = (\hat{\eta_i}, \hat{\eta_j})_{L^2(U)} = (H_i^0, H_j^0)_{L^2(U)} = \delta_{i,j} \quad , \qquad i, j = 1, 2, \cdots, d_N.$$

So for any $q_N \in V_N$ with $q_N = \sum_{i=1}^{d_N} \alpha_i \eta_i$, there is a unique

$$\alpha = (\alpha_1, \cdots \alpha_{d_N}) \in R^{d_N}$$

and

$$|\alpha| = \sqrt{\sum_{i=1}^{d_N} |\alpha_i|^2} = \sqrt{\alpha^T \alpha}$$

In addition,

$$\|q_N\|^2_{*L^2(\Gamma)} = (q_N, q_N)_{*L^2(\Gamma)} = (\sum \alpha_i\eta_i, \sum \alpha_i\eta_i)_{*L^2(\Gamma)} = \sum_{i=1}^{d_N}\sum_{j=1}^{d_N} \alpha_i\alpha_j(\eta_i,\eta_j)_{*L^2(\Gamma)} = \alpha^T\alpha$$

From Lemma 2.5, there are constants C_1, C_2 (independent of N) for which

$$C_1\|q_N\|_{*L^2(\Gamma)} \leq \|q_N\|_{L^2(\Gamma)} \leq C_2\|q_N\|_{*L^2(\Gamma)}, \quad N \geq 1$$

So

$$(1) \qquad C_1\alpha^T\alpha \leq \|q_N\|^2_{L^2(\Gamma)} \leq C_2\alpha^T\alpha$$

From previous results (in the proof of Theorem 2.10), we have

$$a(q_N, q_N) \geq C_3\|q_N\|^2_{H^{-\frac{1}{2}}(\Gamma)}$$

and

$$\|q_N\|_{L^2(\Gamma)} = \|L_N q\|_{L^2(\Gamma)} \leq C_4 N^{\frac{1}{2}}\|L_N q\|_{H^{-\frac{1}{2}}(\Gamma)} = C_4 N^{\frac{1}{2}}\|q_N\|_{H^{-\frac{1}{2}}(\Gamma)}$$

with C_3, C_4 are constant. So

$$a(q_N, q_N) \geq C_3\|q_N\|^2_{H^{-\frac{1}{2}}(\Gamma)} \geq C_5 N^{-1}\|q_N\|^2_{L^2(\Gamma)}$$

with C_5 constant, and

$$\begin{aligned} a(q_N, q_N) &= (\mathcal{K}q_N, q_N) \\ &\leq \|\mathcal{K}q_N\|_{H^{\frac{1}{2}}(\Gamma)}\|q_N\|_{H^{-\frac{1}{2}}(\Gamma)} \\ &\leq \|\mathcal{K}\|\|q_N\|^2_{H^{-\frac{1}{2}}(\Gamma)} \leq C_6\|q_N\|^2_{L^2(\Gamma)} \end{aligned}$$

where C_6 is a constant. Thus

$$(2) \qquad C_5 N^{-1}\|q_N\|^2_{L^2(\Gamma)} \leq a(q_N, q_N) \leq C_5\|q_N\|^2_{L^2(\Gamma)}$$

Hence we have the following from (1) and (2):

$$\frac{\alpha^T B\alpha}{\alpha^T\alpha} = \frac{a(q_N, q_N)}{\alpha^T\alpha} \leq \frac{C_6\|q_N\|^2_{L^2(\Gamma)}}{\alpha^T\alpha} \leq C$$

$$\frac{\alpha^T B\alpha}{\alpha^T\alpha} = \frac{a(q_N, q_N)}{\alpha^T\alpha} \geq \frac{C_5 N^{-1}\|q_N\|^2_{L^2(\Gamma)}}{\alpha^T\alpha} \geq C\frac{1}{N}$$

Then

$$\lambda_{max} = \sup_{\alpha\neq 0}\frac{\alpha^T B\alpha}{\alpha^T\alpha} \leq C$$

$$\lambda_{min} = \inf_{\alpha\neq 0}\frac{\alpha^T B\alpha}{\alpha^T\alpha} \geq C\frac{1}{N}$$

Hence

$$Cond(B) = \frac{\lambda_{max}}{\lambda_{min}} \leq CN$$

for some constant C.

□

2.4 Error bound for u_N

The error bound for solving (2) is obtained as follow. Let

$$u_N(X) = \frac{1}{4\pi}\int_\Gamma \frac{q_N(Y)}{|X-Y|}dY, \quad X \in \Omega \tag{11}$$

Using the maximum principle for harmonic functions,

$$\max_{X\in\Omega}|u(X) - u_N(X)| = \max_{Z\in\Gamma}|u(Z) - u_N(Z)|$$

with respect to the true solution u of (2). Let $X \to Z \in \Gamma$ from (11), we have

$$u_N(Z) = \mathcal{K}q_N(Z)$$

So

$$|u(z) - u_N(Z)| = |\mathcal{K}(q - q_N)(Z)|$$

Recall that for any $0 < \epsilon \leq 1$, $H^{1+\epsilon}(\Gamma) \subset C(\Gamma)$ and

$$\|g\|_\infty \leq \|g\|_{H^{1+\epsilon}}$$

Also

$$\mathcal{K} : H^\epsilon \to H^{1+\epsilon}$$

is a bounded operator, see [10, p.124]. Thus for $Z \in \Gamma$,

$$\begin{aligned} |\mathcal{K}(q - q_N)(Z)| &\leq C\|\mathcal{K}(q - q_N)\|_{H^{1+\epsilon}} \\ &\leq C\|q - q_N\|_{H^\epsilon} \end{aligned}$$

By Theorem 2.9

$$\|q - q_N\|_{H^\epsilon} \leq CN^{\epsilon - r}\|q\|_{H^r}$$

for any $q \in H^r, \quad r \geq \epsilon$.
Hence

$$\max_{X \in \Omega} |u(X) - u_N(X)| \leq CN^{\epsilon - r}\|q\|_{H^r(\Gamma)} \tag{12}$$

and this converges to zero as $N \to \infty$, provided $q \in H^r(\Gamma)$ with $r > \epsilon$.

3 Implementation of Galerkin's method

The Galerkin's method we described above converts our problem (3) to the usually small linear system (7). The most difficult part of the implementation is the calculation of the Galerkin coefficients $(\eta_i, \mathcal{K}\eta_j)$, for $1 \le i, j \le d_N$. Each of these is a double surface integral over Γ, with $\mathcal{K}\eta_i$ involving a singular integrand; and both integrals must be calculated numerically. With a proper transformation (see [2, p.4]), we can assume that all integrals are over U, the unit sphere. We use a 'product Gaussian quadrature method', which is discussed in paper [3].

Let

$$I(f) = \int_U f(Q) ds_Q = \int_0^{2\pi} \int_0^{\pi} f(\phi, \theta) \sin(\theta) d\theta d\phi$$

with $f(\phi, \theta)$ denoting $f(Q)$ with $Q = (\cos\phi \sin\theta, \sin\phi \sin\theta, \cos\theta)$. Approximate this by the product Gaussian formula

$$I_M(f) = \delta \sum_{i=1}^{2M} \sum_{j=1}^{M} \omega_j f(\phi_i, \theta_j) \tag{13}$$

Here $M \ge 1$, $\delta = \frac{\pi}{M}$, $\phi_i = i\delta$ for $i = 1, 2, \cdots 2M$; and $\{\omega_j\}$, $\{\cos(\theta_j)\}$ are the Gauss-Legendre weights and nodes of order M on [-1,1]. The degree of precision of the formula is $2M - 1$ (see Stroud [13, p. 40], or see [3]).

Theorem 3.11 *For $f \in H^t(U)$ with $t > 1$, $t \in \mathbf{R}$,*

$$I_M(f) \to I(f)$$

as $M \to \infty$. Moreover,

$$|I(f) - I_M(f)| \le \frac{C}{(4M+1)^{t-1}} \|f\|_{H^t(U)}.$$

Proof: For $t > 1$, $H^t(U) \subset C(U)$ From [3, Theorem 3],

$$I_M(f) \to I(f) \ \text{ as } \ M \to \infty$$

for $f \in C(U)$. Also, we can write $f \in H^t(U)$ as

$$f = \sum_{n=0}^{\infty} \left[a_n H_n + \sum_{m=1}^{n} (a_n^m H_{n1}^m + b_n^m H_{n2}^m) \right] \tag{14}$$

where $H_n, H_{n1}^m, H_{n2}^m, \quad m = 1, \cdots n, n = 0, \cdots \infty$ are orthonormal basis from the normalization of orthogonal basis functions

$$P_n(\cos\theta), P_n^m(\cos\theta)\cos(m\phi), P_n^m(\cos\theta)\sin(m\phi)$$

for $1 \leq m \leq n, n = 0, 1, \cdots \infty$.

Using these basis functions, we can write

$$\begin{aligned} f(\phi,\theta) &= \sum_{n=0}^{\infty}\left[A_n P_n(\cos\theta) + \sum_{m=1}^{n}\{A_n^m\cos(m\phi) + B_n^m\sin(m\phi)\}P_n^m(\cos\theta)\right] \\ &= g(\phi,\theta) + \sum_{n=2M}^{\infty}\left[A_n P_n(\cos\theta) + \sum_{m=1}^{n}\{A_n^m\cos(m\phi) + B_n^m\sin(m\phi)\}P_n^m(\cos\theta)\right] \end{aligned}$$

where $\{A_n^m, B_n^m\}$ are given below in (15) and (16), and each such coefficient is an appropriate multiple of the corresponding coefficient in (14). Also, $g(\phi,\theta)$ is a polynomial degree $\leq 2M-1$,

$$g(\phi,\theta) = \sum_{n=0}^{2M-1}\left[A_n P_n(\cos\theta) + \sum_{m=1}^{n}\{A_n^m\cos(m\phi) + B_n^m\sin(m\phi)\}P_n^m(\cos\theta)\right]$$

Since

$$\int_U p_n^m(cos\theta) sin m\phi dU = 0$$

$$\int_U p_n^m(cos\theta) cos m\phi dU = 0$$

$$\int_U p_n(cos\theta) dU = 0$$

for $n \geq 1$. Hence

$$\begin{aligned} I(f) &= I(g) + \sum_{n=2M}^{\infty}[\int_0^{2\pi}\int_0^{\pi} A_n P_n(\cos\theta)\sin\theta d\theta d\phi \\ &+ \sum_{m=1}^{n}\{A_n^m \int_0^{2\pi}\int_0^{\pi}\cos(m\phi)P_n^m(\cos\theta)\sin\theta d\theta d\phi \\ &+ B_n^m\int_0^{2\pi}\int_0^{\pi}\sin(m\phi)P_n^m(\cos\theta)\sin(\theta)d\theta d\phi\}] \\ &= I(g) \end{aligned}$$

For the numerical integral $I_M(f)$,

$$\begin{aligned} I_M(f) &= I_M(g) + \sum_{n=2M}^{\infty} [A_n I_M(P_n(\cos\theta)) \\ &+ \sum_{m=1}^{n} \{A_n^m I_M(\cos m\phi P_n^m(\cos\theta)) + B_n^m I_M(\sin m\phi P_n^m(\cos\theta))\}] \end{aligned}$$

So we have

$$\begin{aligned} I(f) - I_M(f) &= -\sum_{n=2M}^{\infty} [A_n I_M(P_n(\cos\theta)) \\ &+ \sum_{m=1}^{n} \{A_n^m I_M(\cos m\phi P_n^m(\cos\theta)) + B_n^m I_M(\sin m\phi P_n^m(\cos\theta))\}] \end{aligned}$$

By [7, pp. 88,118],

$$\int_{-1}^{1} P_n^2(x)dx = \frac{2}{2n+1}$$

$$\int_{-1}^{1} |P_n^m(x)|^2 dx = \frac{(n+m)!}{(n-m)!}\frac{2}{2n+1}$$

Let

$$A_n^0 = A_n, \quad a_n^0 = a_n$$

Then

$$A_n^m = \sqrt{\frac{2n+1}{2}\frac{(n-m)!}{(n+m)!}}\ a_n^m \quad \text{for} \ \ m = 0, \cdots n \tag{15}$$

$$B_n^m = \sqrt{\frac{2n+1}{2}\frac{(n-m)!}{(n+m)!}}\ b_n^m \quad \text{for} \ \ m = 1, \cdots n \tag{16}$$

The trapezoidal rule is given by

$$\hat{T_m}(f) = h\sum_{j=0}^{m-1} f(t_j)$$

with $h = \frac{2\pi}{m}$, $t_j = jh$, for $0 \le j < \infty$. We have

$$\hat{T_m}(e^{ikt}) = \begin{cases} 2\pi & , \quad k = 0 \pmod{m} \\ 0 & , \quad k \ne 0 \pmod{m} \end{cases}$$

see [5, p. 17]. So $\hat{T_m}(\sin kt) = 0$, for all k, that is

$$h \sum_{j=0}^{m-1} \sin(kt_j) = 0$$

for any k. Hence we have the following:

$$\begin{aligned} I_M(\sin(m\phi)P_n^m(\cos(\theta))) &= (\delta \sum_{j=0}^{2M-1} \sin(mj\delta))(\sum_{i=1}^{M} \omega_i P_n^m(\cos\theta_i)) \\ &= 0 \end{aligned}$$

for all integer m.

Since $\|I_M\| = 4\pi$ as a linear function on $C(U)$, we have

$$\delta \sum_{i=1}^{2M} \sum_{j=1}^{M} \omega_j = 4\pi$$

Then

$$\begin{aligned} |I(f) - I_M(f)| &= \left| \delta \sum_{i=1}^{2M} \sum_{j=1}^{M} \omega_j \sum_{n=2M}^{\infty} \sum_{m=0}^{n} A_n^m P_n^m(\cos\theta_j) \cos m\phi_i \right| \\ &= \left| \delta \sum_{i=1}^{2M} \sum_{j=1}^{M} \omega_j \right| \left| \sum_{n=2M}^{\infty} \sum_{m=0}^{n} \sqrt{\frac{2n+1}{2} \frac{(n-m)!}{(n+m)!}} a_n^m \cos m\phi_i P_n^m(\cos\theta_j) \right| \\ &\leq \delta \sum_{i=1}^{2M} \sum_{j=1}^{M} \omega_j \sum_{n=2M}^{\infty} \sum_{m=0}^{n} \sqrt{\frac{2n+1}{2} \frac{(n-m)!}{(n+m)!}} |P_n^m(\cos(\theta_j))| \; |a_n^m| \\ &\leq 4\pi \sum_{n=2M}^{\infty} \left[\sum_{m=0}^{n} \frac{2n+1}{2} \frac{(n-m)!}{(n+m)!} \; |P_n^m(\cos\theta_j)|^2 \right]^{\frac{1}{2}} \left[\sum_{m=0}^{n} |a_n^m|^2 \right]^{\frac{1}{2}} \\ &= 4\pi \sum_{n=2M}^{\infty} \sqrt{\frac{2n+1}{2}} \; \left[\sum_{m=0}^{n} |a_n^m|^2 \right]^{\frac{1}{2}} \end{aligned}$$

This last step uses the result from [8, p. 7]. Then

$$|I(f) - I_M(f)| \leq 4\pi \sum_{n=2M}^{\infty} \left[\frac{(2n+1)^{-2t+1}}{2} \sum_{m=0}^{n} (2n+1)^{2t} |a_n^m|^2 \right]^{\frac{1}{2}}$$

$$
\begin{aligned}
&\leq 4\pi \left[\sum_{n=2M}^{\infty} \frac{(2n+1)^{-2t+1}}{2} \right]^{\frac{1}{2}} \left[\sum_{n=2M}^{\infty} \sum_{m=0}^{n} (2n+1)^{2t} |a_n^m|^2 \right]^{\frac{1}{2}} \\
&\leq C \left[\int_{2M}^{\infty} (2x+1)^{-2t+1} dx \right]^{\frac{1}{2}} \|f\|_{H^t(U)} \\
&\leq \frac{C}{(4M+1)^{t-1}} \|f\|_{H^t(U)}
\end{aligned}
$$

where C is a constant depending on t. □

Note: The result obtained here is slightly stronger than the convergence result from [3, p. 334].

3.1 Numerical Evaluation of Galerkin Coefficients and u_N

To evaluate $\mathcal{K}\eta_j$, we use a method described in [4]. Write

$$
\mathcal{K}\eta_j = \hat{\mathcal{K}}\hat{\eta}_j = \hat{\mathcal{K}}H_j^0 = \int_U H_j^0(Q_u)\hat{K}(P_u, Q_u) ds_{Q_u} \tag{17}
$$

with $P_u \in U$ and an appropriate $\hat{\mathcal{K}}$ defined on U, $\hat{K}$ is the kernel function for mapping $\hat{\mathcal{K}}$. Note this integrand has an unbounded discontinuity in $\hat{K}(P_u, Q_u)$ at $Q_u = P_u$, If we apply method (13) to (17) directly, the convergence will be slow. So in order to avoid this, we use a new spherical coordinates representation for the integral, see [4, pp. 89-90], with

$$
Q_u = Q_u(\phi'', \theta'') = (\cos\phi'' \sin\theta'', \sin\phi'' \sin\theta'', \cos\theta'')
$$

with $Q_u = P_u$ corresponding to $\theta'' = 0$ or $\theta'' = \pi$. Then

$$
\begin{aligned}
\int_U H_j^0(Q_u)\hat{K}(P_u, Q_u) ds_{Q_u} &= \int_0^{2\pi} \int_0^{\pi} H_j^0(Q_u(\phi'', \theta''))\hat{K}(P_u, Q_u(\phi'', \theta'')) \sin\theta'' d\theta'' d\phi'' \\
&= \int_0^{2\pi} \int_{-1}^{1} H_j^0(Q_u(\phi'', \cos^{-1} z))\hat{K}(P_u, Q_u(\phi'', \cos^{-1} z)) dz d\phi''
\end{aligned}
$$

The singularity in the integrand now occurs along the line $z = -1$ or $z = 1$.

The convergence for applying the method (13) to the above integral is quite good in our examples. For the entire coefficient, we use

$$
(\eta_i, \mathcal{K}\eta_j) = (H_i^0, \hat{\mathcal{K}}H_j^0) \approx I_M(H_i^0, \hat{\mathcal{K}}H_j^0)
$$

and we use the standard rule I_M for the inner product integral. Since $\hat{\mathcal{K}}H_j^0$ is quite smooth, the method is quite accurate with a smaller $M = M_o$(outer integral parameter) than is used for the singular integration(inner integral parameter) $M = M_i$. In general, we choose $M > N$ in our examples. For more details, see [3, pp. 89-90].

The integral for $u_N(X)$, $X \in \Omega$, is evaluated using I_M to approximate (11). Note that when $X \notin \Gamma$, the integration kernel is smooth; but when X approaches the boundary, the integrand is increasingly peaked. This means that numerical integration with a fixed M is less accurate as X approaches Γ. This is shown in later examples.

4 Numerical examples

We use the algorithm described in [4] and we choose three kinds of surfaces in R^3. The true solutions are from the following:

$$\begin{aligned} u^1 &= 1 \\ u^2 &= x \\ u^3 &= x^2 + y^2 + z^2 \\ u^4 &= e^x \cos y + e^z \sin x \\ u^5 &= ((x-5)^2 + (y-4)^2 + (z-3)^2)^{-\frac{1}{2}} \end{aligned}$$

EXAMPLE 1 Let $\Gamma = U$, the unit sphere, $x^2+y^2+z^2 = 1$. We calculate the error $u - u_4$, in which the degree of the approximating polynomial is 4, the integration parameters for calculating the Galerkin coefficient are: inner integration parameter $M_i = 32$ and outer integration parameter $M_o = 16$. In Table 1, the first error column is based on the boundary function u^1, and the second, on u^5. Both have an accurate solution, and we do not need a higher degree of approximation. Note that for $u = u^1$, the errors in the table are entirely due to the numerical integration of the various inner products, and the numerical integration of u^4 as defined by (13).

Table 1. $\Gamma =$ Unit sphere

	x	y	z	$u^1 - u^1_4$	$u^5 - u^5_4$
	0.0000	0.0000	0.0000	$-1.408D-06$	$-1.992D-07$
	0.1000	0.1000	0.1000	$-1.408D-06$	$-2.131D-07$
	0.2500	0.2500	0.2500	$-1.408D-06$	$-1.465D-07$
	0.5000	0.5000	0.5000	$-2.507D-06$	$2.602D-06$

EXAMPLE 2 Let Γ be the ellipsoid

$$x^2 + \left(\frac{y}{1.5}\right)^2 + \left(\frac{z}{2}\right)^2 = 1 \quad \text{(E1)}$$

We still use $M_i = 32,\ M_o = 16$ for calculating the Galerkin coefficients. First, we calculate the Laplace expansion for $u^5(\bar{M}^{-1}(X))$, for $X \in \Gamma$, on unit sphere, to see how the Laplace coefficients decrease as N increases. See Table 2.

Table 2. Maximum Laplace coefficients for $u = u^5$ with $N = 8$

degree I	the maximum coefficient for degree I
0	$1.1082094548D + 00$
1	$6.6129416332D - 02$
2	$1.3768464426D - 01$
3	$-1.1407979356D - 02$
4	$8.6900303697D - 03$
5	$4.5775822458D - 04$
6	$1.3675609828D - 03$
7	$8.3698632603D - 05$
8	$2.6810746805D - 04$

Second, we want to calculate $u_N(X),\ X \in \Omega$. When X approaches the boundary Γ, the integrand becomes increasingly ill-behaved; and we need to increase the degree N of the approximation and the integration parameter(say M) for getting $u_N(X)$, in order to have sufficient accuracy near the boundary. For this example, the point with the largest error at which we evaluated $u_N(X)$ was $(0.7, 0.7, 0.7)$. In Table 3, the first error column is based on $N = 4$ with an integration parameter $M = 32$, and the second error column is based on $N = 6, M = 64$.

Table 3 shows an increase in accuracy when the approximation degree is increased. For different functions u, we need different degrees in order to have comparable sized error. This can be seen in comparing Table 3 and Table 4 Finally, we show in Figure 1 how the error behaviour varies with the approximation degree. In the graph, the solid line '-' is the maximum of the absolute value of the errors among the 4 points $(0, 0, 0)$,$(0.1, 0.1, 0.1)$,$(0.25, 0.25, 0.25)$ and $(0.5, 0.5, 0.5)$ for $u = u^5$; the star line '*' is that for $u = u^1$; and the circle line 'o', is that for $u = u^3$.

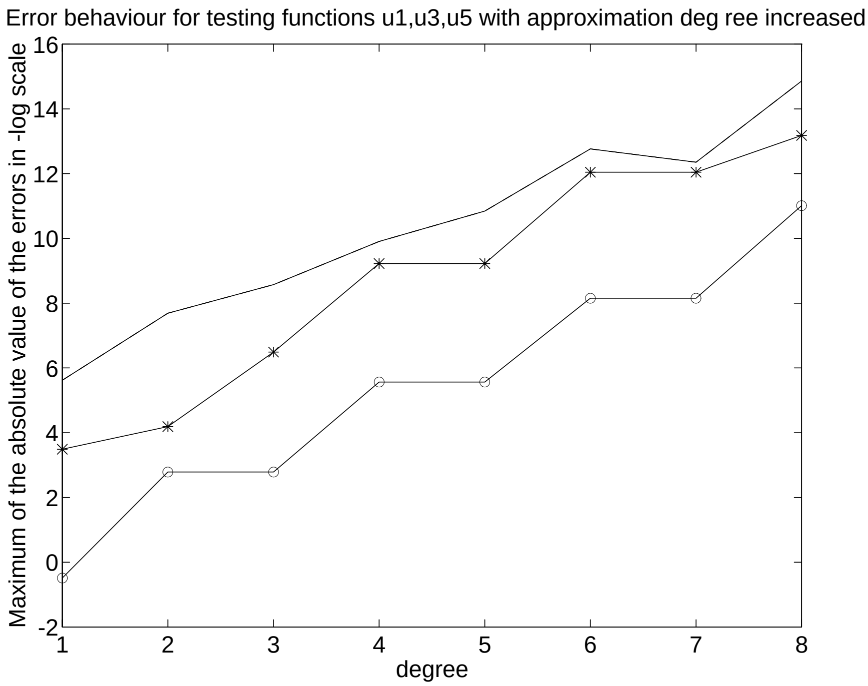

Figure 1

Table 3. The error for the case of $u = u^1$ on the ellipsoid

	x	y	z	$\mathrm{u}^1 - \mathrm{u}^1_4$	$\mathrm{u}^1 - \mathrm{u}^1_6$
	0.00	0.00	0.00	$-9.876D - 05$	$5.878D - 06$
	0.10	0.10	0.10	$-9.844D - 05$	$5.809D - 06$
	0.25	0.25	0.25	$-8.664D - 05$	$3.337D - 06$
	0.50	0.50	0.50	$8.048D - 05$	$-2.641D - 05$
	0.70	0.70	0.70	$7.435D - 04$	$-7.526D - 05$

Table 4. The error for the case $u = u^3$ and $u = u^5$ on the ellipsoid

	x	y	z	$\mathrm{u}^3 - \mathrm{u}_9^3$	$\mathrm{u}^5 - \mathrm{u}_7^5$
	0.00	0.00	0.00	$1.652D - 05$	$7.107D - 07$
	0.10	0.10	0.10	$1.606D - 05$	$7.263D - 07$
	0.25	0.25	0.25	$-9.898D - 07$	$2.301D - 07$
	0.50	0.50	0.50	$-2.080D - 04$	$-4.309D - 06$

If we define Γ to be the ellipsoid

$$x^2 + \left(\frac{y}{2}\right)^2 + \left(\frac{z}{5}\right)^2 = 1 \quad \text{(E2)}$$

it is more ill-behaved than E1 in the sense of being more thin and narrow in shape. Consequently the accuracy is worse for comparable values of N. This is shown in Table 5 which uses $N = 7$, $M = 32$.

Table 5. Errors for ellipsoid (E2)

	x	y	z	$\mathrm{u}^3 - \mathrm{u}_7^3$	$\mathrm{u}^5 - \mathrm{u}_7^5$
	0.00	0.00	0.00	$-2.067D - 01$	$1.686D - 04$
	0.10	0.10	0.10	$-2.066D - 01$	$1.788D - 04$
	0.25	0.25	0.25	$-2.056D - 01$	$1.876D - 04$
	0.50	0.50	0.50	$-2.073D + 01$	$1.834D - 04$

Now if we increase the degree of the approximation ($N = 9$) and the integration parameter($M = 64$) for calculating $u_N(A)$, $A \in \Omega$, we obtain better results, as shown in Table 6.

Table 6. Improved errors for ellipsoid (E2)

	x	y	z	$\mathrm{u}^3 - \mathrm{u}_9^3$	$\mathrm{u}^5 - \mathrm{u}_9^5$
	0.00	0.00	0.00	$6.650D - 02$	$-5.576D - 05$
	0.10	0.10	0.10	$6.648D - 02$	$-5.988D - 05$
	0.25	0.25	0.25	$6.559D - 02$	$-6.356D - 05$
	0.50	0.50	0.50	$5.272D - 02$	$-5.259D - 05$

EXAMPLE 3 We use a peanut-shaped region, based on the ovals of Cassini. Γ is defined by

$$R = \sqrt{\cos(2\theta) + \sqrt{\alpha + 1 - \sin(2\theta)^2}}$$

$$(x, y, z) = R(\sin\theta\cos\phi, 2\sin\theta\sin\phi, \cos\theta)$$

The larger α is, the better is the behaviour of the solution procedure, based on comparing the results in Table 7 and Table 8. Figures 2 and 3 are the cross-sections in the xz-plane with $\alpha = 0.8$ and $\alpha = 0.1$ respectively. Also, for both the $\alpha = 0.8$ and $\alpha = 0.1$ cases, function u^5 gives smaller errors than function u^4 does, since u^4 varies greatly and more rapidly over some sections of this region.

Table 7. $\alpha = 0.8$ for "peanut" region

	x	y	z	$\mathrm{u}^4 - \mathrm{u}^4_8$	$\mathrm{u}^5 - \mathrm{u}^5_8$
	0.00	0.00	0.00	$-8.958D-06$	$3.566D-07$
	0.10	0.10	0.10	$-2.554D-05$	$2.882D-07$
	0.25	0.25	0.25	$-1.613D-04$	$-1.130D-06$
	0.00	0.00	0.30	$-1.634D-05$	$7.020D-07$

Table 8. $\alpha = 0.1$ for "peanut" region

	x	y	z	$u^4 - u^4_9$	$u^5 - u^5_9$
	0.00	0.00	0.00	$-7.256D-05$	$-3.609D-06$
	0.00	0.00	0.40	$-2.041D-04$	$-1.313D-05$
	0.80	0.80	0.80	$-4.001D-04$	$7.169D-05$
	0.03	0.06	0.04	$-8.324D-05$	$-3.498D-06$
	0.07	0.14	0.10	$-1.494D-04$	$-4.669D-06$

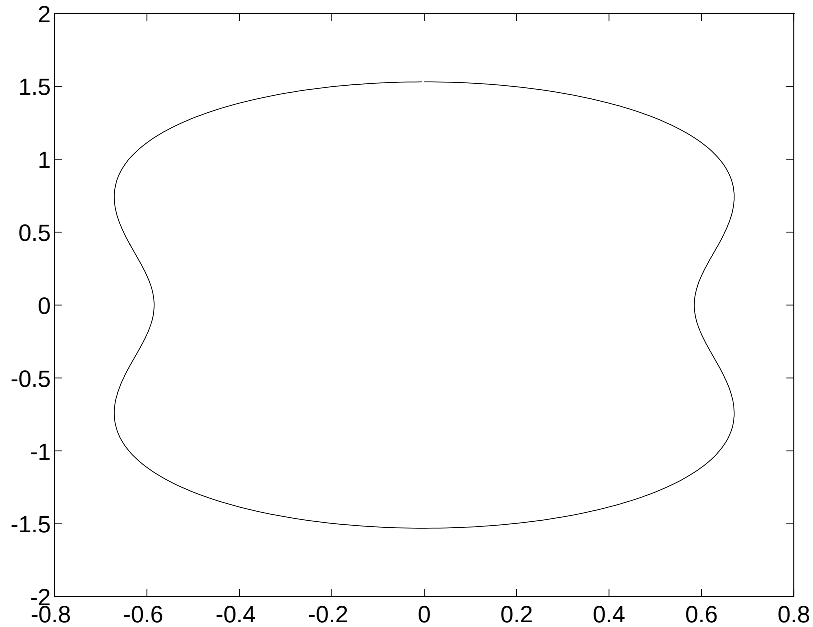

Figure 2

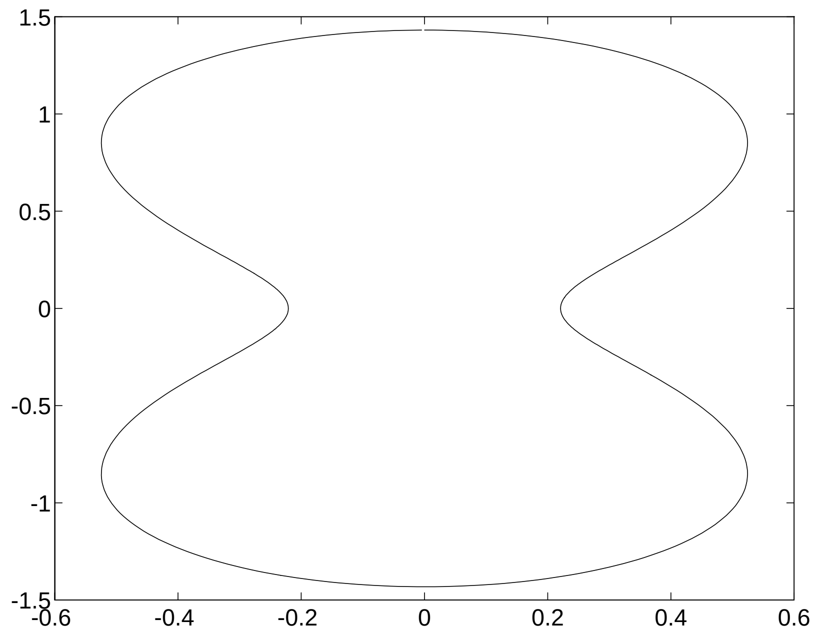

Figure 3

EXAMPLE 4 Finally, we define a heart-shaped region:

$$(x, y, z) = R(2 \sin\theta \cos\phi, \sin\theta \cos\phi, \cos\theta)$$

$$R(x, y, z) = 2 - (1 + 100(z+1)^2)^{-1}, \quad (x, y, z) \in U$$

In Table 9, the first error column is based on using the boundary function u^1; and the second and third error columns are based on u^5 with $N = 4$ and $N = 8$ respectively. Once we choose sufficiently large inner and outer integration parameters M_i and M_o, for calculating the Galerkin coefficients, (for example: $M_i = 32$ and $M_o = 16$),the accuracy of the error depends on the approximation degree N.

Table 9. Using different boundary functions for testing error

	x	y	z	$\mathrm{u}^1 - \mathrm{u}^1_8$	$\mathrm{u}^5 - \mathrm{u}^5_4$	$\mathrm{u}^5 - \mathrm{u}^5_8$
	0.10	0.10	0.10	$-8.724D-05$	$-1.159D-04$	$-6.187D-06$
	2.00	0.00	0.00	$-2.370D-05$	$2.891D-05$	$-2.728D-06$
	0.00	1.00	0.00	$-9.478D-05$	$-2.213D-05$	$-6.197D-06$
	2.00	0.00	1.00	$-2.903D-05$	$-1.516D-04$	$-4.301D-06$

References

[1] K. E. Atkinson(1980) The numerical solution of Laplace's equation in three dimensions - II, in *Numerical Treatment of Integral Equations*, ed. by J. Albrecht and L. Collatz, Birkhauser, Basel, pp. 1-23.

[2] K. E. Atkinson(1982) The numerical solution of Laplace's equation in three dimensions, *SIAM. J. on Numer. Anal.* **19**, pp. 263-274.

[3] K.E.Atkinson(1982) Numerical integration on the sphere, *J. Austral. Math. Soc.(Series B)*, **23**, pp. 332-347.

[4] K. E. Atkinson(1985) Algorithm 629: An integral equation program for Laplace's equation in three dimensions, *ACM Transactions on Mathematical Software*, **11**, pp. 85-96.

[5] Kendall E. Atkinson and Ian H. Sloan(1991) The numerical solution of first-kind logarithmic-kernel integral equations on smooth open arcs, *Math. of Computation* **56**, pp. 119-139.

[6] J. C. Nedelec(1976) Curved finite element methods for the solution of singular integral equations on surfaces in R^3, *Computer Methods in Applied Mechanics and Engineering* **8**, pp. 61-80.

[7] T. M. MacRobert(1967) *Spherical Harmonics*, Pergamon Press, London.

[8] L. Wienert(1990) *Die numerische Approximation von Randintegraloperatoren fur die Helmholtzgleichung im* R^3, Dissertation, der Georg-August-Universität zu Göttingen.

[9] P. Grisvard(1985) *Elliptic Problems in Nonsmooth Domains*, Pitman Press.

[10] Robert Dautray and Jacques-Louis Lions (1984) *Mathematical Analysis and Numerical Methods for Science and Technology*, Vol.**4**, Integral Equations and Numerical Methods, Springer-Verlag, Berlin.

[11] Soloman G. Mikhlin and Siegfried Prossdorf(1986) *Singular Integral Operators*, Springer-Verlag, Berlin.

[12] Claes Johnson(1987) *Numerical Solution of Partial Differential Equations by the Finite Element Method*, Cambridge University Press, Cambridge.

[13] A.H.Stroud(1971) *Approximate Calculation of Multiple Integrals*, Prentice-Hall, Englewood Cliffs, NJ.

[14] Philippe G. Ciarlet(1978) *The Finite Element Method for Elliptic Problems*, Amsterdam, New York, Oxford.