A Personal Perspective on the History of the Numerical Analysis of Fredholm Integral Equations of the Second Kind

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Abstract

This is a personal perspective on the development of numerical methods for solving Fredholm integral equations of the second kind, discussing work being done principally during the 1950s and 1960s. The principal types of numerical methods being studied were projection methods (Galerkin, collocation) and Nyström methods. During the 1950s and 1960s, functional analysis became the framework for the analysis of numerical methods for solving integral equations, and this influenced the questions being asked. This paper looks at the history of the analyses being done at that time.

1 Introduction

This memoir is about the history of the numerical analysis associated with solving the integral equation

$$\lambda x(s) - \int_a^b K(s,t)x(t)\,dt = y(s), \qquad a \le s \le b, \quad \lambda \neq 0 \tag{1}$$

At the time I was in graduate school in the early 1960's, researchers were interested principally in this one-dimensional case. It was for a kernel function K that was at least continuous; and generally it was assumed that $K(s,t)$ was several times continuously differentiable. This was the type of equation studied by Ivar Fredholm [26], and in his honor such equations are called *Fredholm integral equations of the second kind*.

Today we work with multi-dimensional Fredholm integral equations of the second kind in which the integral operator is completely continuous and the integration region is commonly a surface in $\mathbb{R}^3$; in addition, the kernel function K is often discontinuous. The Fredholm theory is still valid for such equations, and this theory is critical for the convergence and stability analysis of associated

numerical methods. Throughout this paper, we assume the integral equation (1) is uniquely solvable for any given continuous function y.

The theory of Fredholm integral equations is quite old, and many such equations are associated with reformulations of elliptic boundary value problems as *boundary integral equations* (BIEs). More about BIEs later. Among the well-known names associated with the development of the theory of Fredholm integral equations are Ivar Fredholm and David Hilbert. For a survey of the origins of integral equations in applications, see Lonseth [44]. An interesting history of the origins of function spaces is given in Bernkopf [16]. He argues that much of the original motivation for creating function spaces arises from the study of integral equations (and secondarily, the calculus of variations).

A brief perusal of any mathematics library will yield many books on integral equations. An excellent introductory text on integral equations is that of Rainer Kress [41], and it also contains a good introduction to the numerical solution of integral equations.

2 A survey of numerical methods

There are only a few books on the numerical solution of integral equations as compared to the much larger number that have been published on the numerical solution of ordinary and partial differential equations. General books on the numerical solution of integral equations include, in historical order, Bückner [21], Baker [14], Delves and Mohamed [25], and Hackbusch [32]. Lonseth [43] gives a survey of numerical methods in the period preceding the widespread use of digital computers. For an interesting perspective on the theory, application, and numerical solution of nonlinear integral equations around the early 1960's, see the proceedings [1]. Important bibliographic references for both the application and the solution of integral equations are given in the very large subject and author listings of Noble [46].

Bückner's book was published in 1952, and it is representative of a pre-computer approach to the numerical analysis of integral equations. The book presents numerical methods for principally Fredholm integral equations of the second kind, with a shorter discussion of numerical methods for Volterra integral equations. The eigenvalue problem for integral operators is the principal focus of the book, with shorter treatments of some numerical methods for the inhomogeneous equation (1).

More specialized treatments of numerical methods for integral equations are given in the books Atkinson [9], [11], Brunner [20], Chatelin [23], Groetsch [29], Linz [42], Ivanov [35], and Wing [65]. Useful presentations of numerical methods are given in sections of [39, Chap. 14], [38, Chap. 2], [41, Chaps 10-17], and [12, Chaps 12, 13], along with sections of many general texts on integral equations. There are a number of edited proceedings, which we omit here. In the last 25 years, there has been a large amount of activity in numerical methods for solving boundary integral equation reformulations of partial differential equations. Introductions to this topic are given in [11], [32], and [41]. It is discussed briefly

later in this paper.

Before discussing some of the history of the numerical analysis for (1), I give a brief survey of the general numerical methods for solving such integral equations. Most researchers subdivide the numerical methods into the following.

- Degenerate kernel approximation methods
- Projection methods
- Nyström methods (also called quadrature methods)

All of these methods have iterative variants, which I discuss briefly in §3.4. There are other numerical methods, but the above methods and their variants include the most popular general methods.

To expedite the presentation, I often use a functional analysis framework, even though such a presentation arose later in the history of these numerical methods. As an illustration, the integral equation (1) can be written abstractly as $(\lambda I - \mathcal{K})\, x = y$ with $\mathcal{K}$ a compact integral operator on a Banach space $\mathcal{X}$, e.g. $C\,[a,b]$ or $L^2\,(a,b)$.

2.1 Degenerate kernel approximation methods

We say $K\,(s,t)$ is a *degenerate kernel function* if it has the form

$$K\,(s,t) = \sum_{j=1}^{n} \alpha_j\,(s)\,\beta_j\,(t)$$

In this case, the solution of (1) reduces to the solution of the linear system

$$\lambda c_i - \sum_{j=1}^{n} (\alpha_j, \beta_i)\, c_j = (y, \beta_i)\,, \qquad i = 1, \ldots, n$$

and

$$x\,(s) = \frac{1}{\lambda}\left[y\,(s) + \sum_{j=1}^{n} c_j \alpha_j(s) \right] \tag{2}$$

Most kernel functions $K\,(s,t)$ are not degenerate, and thus we seek to approximate them by degenerate kernels. We assume a sequence of degenerate kernels have been constructed, call them $K_n\,(s,t)$, for which

$$\max_{a \le s \le b} \int_a^b |K\,(s,t) - K_n\,(s,t)|\; dt \to 0 \quad \text{as} \quad n \to \infty \tag{3}$$

Denote by x_n the result of solving the integral equation (1) with the approximate kernel K_n replacing K. For later reference, introduce the associated approximating integral operator

$$\mathcal{K}_n z(s) = \int_a^b K_n(s,t) z(t)\, dt, \quad a \le s \le b, \quad z \in \mathcal{X}$$

Usually, $\mathcal{X}$ equals $C[a,b]$ or $L^2(a,b)$. Then x_n satisfies $(\lambda I - \mathcal{K}_n)\, x_n = y$; and if (1) is considered within the framework of the function space $C[a,b]$ with the uniform norm, then (3) is exactly the same as saying

$$\|\mathcal{K} - \mathcal{K}_n\| \to 0 \quad \text{as} \quad n \to \infty \tag{4}$$

2.2 Projection methods

These methods approximate the solution x by choosing an approximation from a given finite dimensional linear subspace of functions, call it $\mathcal{Z}$. Given $z \in \mathcal{Z}$, introduce the residual

$$r = (\lambda I - \mathcal{K})\, z - y$$

We select a particular z, call it x^*, by making the residual r small in some sense. Let $\{\varphi_1, \ldots, \varphi_n\}$ denote a basis for $\mathcal{Z}$. Then we seek

$$x^*(s) = \sum_{j=1}^{n} c_j \varphi_j(s)$$

The residual becomes

$$r(s) = \sum_{j=1}^{n} c_j \{\lambda \varphi_j(s) - \mathcal{K}\varphi_j(s)\} - y(s)$$

- **Collocation method**. Select collocation node points $\{t_1, \ldots, t_n\} \in [a,b]$ and require
$$r(t_i) = 0, \qquad i = 1, \ldots, n$$

- **Galerkin method**. Set to zero the Fourier coefficients of r with respect to the basis $\{\varphi_1, \ldots, \varphi_n\}$,
$$(r, \varphi_i) = 0, \qquad i = 1, \ldots, n$$
The basis $\{\varphi_i\}$ need not be orthogonal. The Galerkin method is also called the *method of moments*.

These are the principal projection methods, although there are others such as the minimization of the L^2 norm of r with respect to the elements in $\mathcal{Z}$.

With both collocation and Galerkin methods, it is possible to define a projection $\mathcal{P}$ with range $\mathcal{Z}$ and for which the numerical method takes the abstract form

$$(\lambda I - \mathcal{P}\mathcal{K})\, x^* = \mathcal{P} y$$

In practice we have a sequence of approximating subspaces $\mathcal{Z} = \mathcal{X}_n$, $n \geq 1$, and associated projections $\mathcal{P}_n$. Thus we have a sequence of approximating equations

$$(\lambda I - \mathcal{P}_n \mathcal{K})\, x_n = \mathcal{P}_n y \tag{5}$$

With Galerkin's method defined on a Hilbert space $\mathcal{X}$, $\mathcal{P}_n x$ is the orthogonal projection of x onto $\mathcal{X}_n$. For the collocation method, $\mathcal{P}_n x$ is the element of $\mathcal{X}_n$ which interpolates x at the node points $\{t_1, \ldots, t_n\}$.

We usually work with cases in which

$$\mathcal{P}_n z \to z \quad \text{as} \quad n \to \infty, \quad \text{for all} \quad z \in \mathcal{X} \tag{6}$$

although there are important cases where this is not satisfied. A weaker but adequate assumption is that the projections satisfy

$$\|\mathcal{K} - \mathcal{P}_n \mathcal{K}\| \to 0 \quad \text{as} \quad n \to \infty \tag{7}$$

This also follows from (6) and the compactness of the operator $\mathcal{K}$. The space $\mathcal{X}$ is generally chosen to be $C[a,b]$ or $L^2(a,b)$, and we are solving (1) for the solution $x \in \mathcal{X}$. For details and examples, see [11, Chap. 3].

Projection methods are probably the most widely used class of methods for solving integral equations. For a presentation and summary of the most recent perspectives on projection methods for solving integral equations of the second kind, see [11, Chap. 3]. This also contains a discussion of 'discrete projection methods' in which integrals in the discretized linear system are replaced by numerical integrals.

2.3 Nyström methods

Approximate the integral operator in (1) using numerical integration. Consider a numerical integration scheme

$$\int_a^b f(t)\,dt \approx \sum_{j=1}^{n} w_j f(t_j)$$

which is convergent as $n \to \infty$ for all continuous functions $f \in C[a,b]$. Then introduce

$$\begin{aligned} \mathcal{K}z(s) &\equiv \int_a^b K(s,t)z(t)\,dt \\ &\approx \sum_{j=1}^{n} w_j K(s,t_j) z(t_j) \equiv \mathcal{K}_n z(s), \qquad a \le s \le b \end{aligned}$$

for all $z \in C[a,b]$. We approximate the equation (1) by

$$(\lambda I - \mathcal{K}_n)\, x_n = y \tag{8}$$

or equivalently,

$$\lambda x_n(s) - \sum_{j=1}^{n} w_j K(s,t_j) x_n(t_j) = y(s), \qquad a \le s \le b \tag{9}$$

This is usually solved by first collocating the equation at the integration node points and then solving the linear system

$$\lambda z_i - \sum_{j=1}^{n} w_j K(t_i, t_j) z_j = y(t_i), \qquad i = 1, \ldots, n \tag{10}$$

in which $z_i \equiv x_n(t_i)$. Originally people would take this solution and then interpolate it in some way so as to extend it to the full interval $[a, b]$. However, it can be shown that the equation (9) furnishes a natural interpolation formula,

$$x_n(s) = \frac{1}{\lambda}\left[y(s) + \sum_{j=1}^{n} w_j K(s, t_j) z_j\right], \qquad a \le s \le b \tag{11}$$

It turns out that this is a very good interpolation formula, as the resulting interpolated values have an accuracy that is comparable to that of the approximate solution $\{z_i\}$ at the integration node points. We solve (10) and may stop there with no interpolation; but for the theoretical analysis of the method, we use (9). This places the original equation (1) and the approximating equation (8) in the same function space, namely $C[a, b]$.

The interpolation formula (11) was noted by Nyström [48]. He was operating in an age of hand calculation, and therefore he wanted to minimize the need for such calculations. The great accuracy of (11) recommended itself, as then one could use a high accuracy integration rule with few nodes (e.g. Gaussian quadrature) while still having an accurate answer over the entire interval $[a, b]$. For that reason, and beginning in [4], I refer to the approximation (9) as the Nyström method. It has also been called the 'method of numerical integration' and the 'analogy method' (cf. [21, p. 105]). As a current example of an actual algorithmic use of Nyström interpolation, see [13].

With the rectangular numerical integration rule, (9) was used by Hilbert [33] in studying the symmetric eigenvalue problem for the integral operator $\mathcal{K}$. He used a limiting procedure to pass from the known properties of the symmetric eigenvalue problem for matrices to results for an integral operator $\mathcal{K}$ with a continuous and symmetric kernel function.

3 Error analysis and some history

The 1960's were a time of major change in numerical analysis, due in large part to the widespread introduction of digital computers. To obtain some sense of the contrast with today, consider my first numerical analysis course in 1961 and the text for it, Hildebrand [34]. This text was well-written, and it was fairly typical of numerical analysis textbooks of that time. The numerical methods were dominated by the need to do hand and desktop calculator computations. There was extensive material on finite differences and on methods that would make use of tables. By the mid-1960's there were several books in which digital computers were now the the main means of implementing methods, and this

in turn led to a different type of numerical scheme. Pre-computer algorithms emphasized the use of tables and the use of the human mind to reduce the need for calculations. The use of computers led to the development of simpler methods in which the calculational power of the computer could profitably be brought to bear. For the numerical solution of integral equations such as (1), a major change was being able to solve much larger systems of linear equations. This had been a major roadblock in the development of numerical methods for integral equations.

A major theme in theoretical numerical analysis in the 1950's and 1960's was the development of general frameworks for deriving and analyzing numerical methods, and such frameworks almost always used the language of functional analysis. This was true in many areas of numerical analysis and approximation theory, although I believe numerical linear algebra was less affected by this focus. Initially researchers were more interested in obtaining a better understanding of existing numerical methods than they were in creating new methods. The development of such abstract frameworks, led to the development of so-called 'optimal' numerical methods. Spline functions and finite element methods are both associated with this search for optimal methods. As the abstract frameworks solidified, they led to the development of new methods for new problems.

Especially important in the building of a more general and theoretical framework was the seminal paper of L. V. Kantorovich [37]. This paper was subsequently translated under the auspices of the U.S. National Bureau of Standards and deseminated fairly widely. The paper was quite long and consisted of several parts. A framework using functional analysis was given for the approximate solution of integral equations and other operator equations. Another part generalized the method of steepest descent to functionals over a Banach space. And yet another part developed a calculus for nonlinear operators on Banach spaces. This was followed by a generalization of Newton's method for solving nonlinear operator equations on Banach spaces. This paper was quite influential on me and many others. It took many years for the ideas in the paper to work their way through the research community. For easier access to the material in [37], see the book of Kantorovich and Akilov [39, Chaps. 14-18]. A related early book of importance for nonlinear integral equations is Krasnoselskii [40].

3.1 Degenerate kernel methods

The error analysis for degenerate kernel methods was well-understood without the need for a functional analysis framework, and carrying it over to function spaces was straightforward. Basically it is a consequence of the *geometric series theorem*. In particular, suppose an operator $\mathcal{A} : \mathcal{X} \to \mathcal{Y}$ is bounded, one-to-one and onto, with $\mathcal{X}$ and $\mathcal{Y}$ Banach spaces. Suppose $\mathcal{B} : \mathcal{X} \to \mathcal{Y}$ is bounded, and further assume that

$$\|\mathcal{A} - \mathcal{B}\| < 1/\left\|\mathcal{A}^{-1}\right\| \tag{12}$$

Then $\mathcal{B}$ is also one-to-one and onto, and its inverse $\mathcal{B}^{-1}$ is bounded. Moreover, $\left\|\mathcal{A}^{-1} - \mathcal{B}^{-1}\right\| = O\left(\|\mathcal{A} - \mathcal{B}\|\right)$. In the case of degenerate kernel methods with

$\mathcal{A} = \lambda I - \mathcal{K}$ and $\mathcal{B} = \lambda I - \mathcal{K}_n$, and working within the context of $C[a,b]$, the bound (4) gives us a bound for $\|\mathcal{A} - \mathcal{B}\|$. More precisely, if

$$\|\mathcal{K} - \mathcal{K}_n\| < \frac{1}{\left\|(\lambda I - \mathcal{K})^{-1}\right\|}$$

then $(\lambda I - \mathcal{K}_n)^{-1}$ exists and it can be bounded uniformly for all sufficiently large n. Letting $(\lambda I - \mathcal{K})\, x = y$ and $(\lambda I - \mathcal{K}_n)\, x_n = y$,

$$\|x - x_n\| \leq \left\|(\lambda I - \mathcal{K}_n)^{-1}\right\| \|\mathcal{K} - \mathcal{K}_n\| \, \|x\|$$

This leads to a straightforward error analysis for the degenerate kernel method when considered within the function space $C[a,b]$ using the uniform norm.

This basic analysis is given in many textbooks when developing the theory of integral equations, although historically if was often without the functional analysis framework; and it was often also used to develop some of the theory of the eigenvalue problem for integral operators. Since the degenerate kernel method was used both as a theoretical tool in developing the theory of integral equations and as a numerical method, it is difficult to give attributions to the development of the numerical method. In talking about the numerical method, much time has been spent on developing various means of approximating general kernel functions $K(s,t)$ with a sequence of degenerate kernels $K_n(s,t)$, and this continues to the present day. For illustrative examples of the degenerate kernel method, see [11, Chap. 2].

3.2 Projection methods

The general framework for projection methods and other approximation methods that was given by Kantorovich [37] was too complicated when considering only projection methods. Later work simplified his framework a great deal, and new perspectives continued to be given well into the 1980's.

The general error analysis for projection methods uses the assumption (7) that $\|\mathcal{K} - \mathcal{P}_n\mathcal{K}\| \to 0$ as $n \to \infty$. With the assumption that $(\lambda I - \mathcal{K})^{-1}$ exists, we write

$$\begin{aligned} \lambda I - \mathcal{P}_n\mathcal{K} &= (\lambda I - \mathcal{K}) + (\mathcal{K} - \mathcal{P}_n\mathcal{K}) \\ &= (\lambda I - \mathcal{K}) \left[I - (\lambda I - \mathcal{K})^{-1} (\mathcal{K} - \mathcal{P}_n\mathcal{K})\right] \end{aligned} \tag{13}$$

With the assumption (7), we have that

$$\left\|(\lambda I - \mathcal{K})^{-1}\right\| \|\mathcal{K} - \mathcal{P}_n\mathcal{K}\| < 1$$

for all sufficiently large n. It then follows from the geometric series theorem that $\left[I - (\lambda I - \mathcal{K})^{-1} (\mathcal{K} - \mathcal{P}_n\mathcal{K})\right]^{-1}$ exists and is bounded, and therefore the same is

true for $(\lambda I - \mathcal{P}_n\mathcal{K})^{-1}$. For the error, let $(\lambda I - \mathcal{K})\, x = y$ and $(\lambda I - \mathcal{P}_n\mathcal{K})\, x_n = \mathcal{P}_n y$. Then

$$x - x_n = \lambda\,(\lambda I - \mathcal{P}_n\mathcal{K})^{-1}\,(x - \mathcal{P}_n x) \tag{14}$$

This implies

$$\frac{|\lambda|}{\|\lambda - \mathcal{P}_n\mathcal{K}\|}\,\|x - \mathcal{P}_n x\| \leq \|x - x_n\| \leq |\lambda|\,\left\|(\lambda - \mathcal{P}_n\mathcal{K})^{-1}\right\|\,\|x - \mathcal{P}_n x\| \tag{15}$$

We have convergence if and only if $\mathcal{P}_n x \to x$ as $n \to \infty$. The speed of convergence of x_n to x is precisely the same as that of $\mathcal{P}_n x$ to x.

A number of researchers have contributed to this theory and to extensions not discussed here. There are many papers on collocation methods for solving Fredholm integral equations of the second kind. For a general framework within a functional analysis framework, I cite in particular those of Phillips [50] and Prenter [51].

3.2.1 Kantorovich and Krylov regularization

An early and interesting variant on projection methods was given by Kantorovich and Krylov [38, p. 150]. As with projection methods, suppose a family of approximating functions $\mathcal{Z}$ is given with basis $\{\varphi_1, \ldots, \varphi_n\}$. Assume a solution for (1) of the form

$$x^*(s) = \frac{1}{\lambda}\left[y(s) + \sum_{j=1}^{n} c_j \varphi_j\,(s)\right] \tag{16}$$

This was motivated, perhaps, by the solution (2) for a degenerate kernel integral equation. In effect we are seeking an approximation of the integral operator term $\mathcal{K}x$ in (1),

The authors looked at the residual r for such an approximating solution and then minimized it in the same manner as with Galerkin's method (although collocation methods can be used equally well). Introduce

$$z^*(s) = \sum_{j=1}^{n} c_j \varphi_j\,(s)$$

substitute it into the formula for $r = (\lambda I - \mathcal{K})\, x^* - y$, and then mimize r with either a Galerkin or collocation method. Then z^* satisfies $(\lambda I - \mathcal{P}\mathcal{K})\, z^* = \mathcal{P}\mathcal{K}y$ and x^* satisfies $(\lambda I - \mathcal{P}\mathcal{K})\, x^* = y$. Because $\mathcal{P}$ has a finite-dimensional range, and because we always assume $\mathcal{P}$ is bounded, the combined operator $\mathcal{P}\mathcal{K}$ can be shown to be an integral operator with a degenerate kernel function. Thus the assumption (16) amounts to the approximation of (1) by a degenerate kernel integral equation.

Another way of picturing this method is to consider (1) in the form

$$x = \frac{1}{\lambda}\,(y + z)\,, \qquad z = \mathcal{K}x \tag{17}$$

The function $\mathcal{K}x$ is often better behaved than the original solution x, and this is particularly true if x is badly behaved (e.g. lacking differentiability at points in $[a, b]$). The function z satisfies the equation

$$(\lambda I - \mathcal{K})\, z = \mathcal{K}y \tag{18}$$

Applying a projection method to this equation and then using (17) leads to the method of Kantorovich and Krylov. The use of the formulation (17)-(18) is often referred to as the Kantorovich and Krylov method of regularizing the integral equation $(\lambda I - \mathcal{K})\, x = y$.

3.2.2 The iterated projection solution

Associated with the projection method solution x_n is the *iterated projection solution*. Given the projection method solution x_n, define

$$\widetilde{x}_n = \frac{1}{\lambda}\left[y + \mathcal{K}x_n\right] \tag{19}$$

Although such iterations are found in the literature in many places, Ian Sloan [54] first recognized the importance of doing one such iteration; and in his honor $\widetilde{x}_n$ is often called the *Sloan iterate*.

The solution $\widetilde{x}_n$ satisfies the equation

$$(\lambda I - \mathcal{K}\mathcal{P}_n)\, \widetilde{x}_n = y \tag{20}$$

and $\mathcal{P}_n\widetilde{x}_n = x_n$. It can be shown that $(\lambda I - \mathcal{K}\mathcal{P}_n)^{-1}$ exists if and only if $(\lambda I - \mathcal{P}_n\mathcal{K})^{-1}$ exists; cf. [11, §3.4].

In the case of the Galerkin method over a Hilbert space $\mathcal{X}$, Sloan showed that the iterated solution $\widetilde{x}_n$ converges to x more rapidly than does the original Galerkin solution x, provided $\mathcal{P}_n$ is pointwise convergent to the identity I on $\mathcal{X}$ (as in (6)). Begin with the identity

$$x - \widetilde{x}_n = (\lambda - \mathcal{K}\mathcal{P}_n)^{-1}\mathcal{K}(I - \mathcal{P}_n)x \tag{21}$$

Note that $I - \mathcal{P}_n$ is a projection, and therefore

$$\begin{aligned}
\mathcal{K}(I - \mathcal{P}_n)x &= \mathcal{K}(I - \mathcal{P}_n)(I - \mathcal{P}_n)x \\
\|\mathcal{K}(I - \mathcal{P}_n)x\| &\le \|\mathcal{K}(I - \mathcal{P}_n)\|\, \|(I - \mathcal{P}_n)x\|
\end{aligned}$$

With Galerkin's method, $I - \mathcal{P}_n$ is an orthogonal projection and is self-adjoint. Also, the norm of an operator on a Hilbert space equals that of its adjoint. Therefore,

$$\begin{aligned}
\|\mathcal{K}(I - \mathcal{P}_n)\| &= \left\|[\mathcal{K}(I - \mathcal{P}_n)]^*\right\| = \|(I - \mathcal{P}_n)^*\mathcal{K}^*\| \\
&= \|(I - \mathcal{P}_n)\mathcal{K}^*\|
\end{aligned}$$

If the operator $\mathcal{K}$ is compact on $\mathcal{X}$, then so is $\mathcal{K}^*$; and when combined with (6), we have $\|(I-\mathcal{P}_n)\mathcal{K}^*\| \to 0$. Completing the error bound,

$$\begin{aligned}\|x-\widetilde{x}_n\| &\leq \left\|(\lambda-\mathcal{K}\mathcal{P}_n)^{-1}\right\| \|\mathcal{K}(I-\mathcal{P}_n)x\| \\ &\leq c\,\|(I-\mathcal{P}_n)\mathcal{K}^*\|\,\|(I-\mathcal{P}_n)x\|\end{aligned}$$

When compared with the earlier result (15), this shows the earlier assertion that the iterated solution $\widetilde{x}_n$ converges to x more rapidly than does the original Galerkin solution x.

For collocation methods, we do not have $\|\mathcal{K}(I-\mathcal{P}_n)\| \to 0$. Nonetheless, the Sloan iterated solution $\widetilde{x}_n$ is still useful. From the property $\mathcal{P}_n\widetilde{x}_n = x_n$, we know that x_n and $\widetilde{x}_n$ agree at the node points $\{t_1,\ldots,t_n\}$. Thus an error bound for $\|x-\widetilde{x}_n\|_\infty$ is also a bound on the error

$$E_n = \max_{1\leq i\leq n} |x(t_i) - x_n(t_i)|$$

To bound E_n, we can use the formula (21) and analyze $\|\mathcal{K}(I-\mathcal{P}_n)x\|_\infty$. Using this, Graeme Chandler in his thesis [22] showed that astute choices of interpolation nodes (e.g. Gauss-Legendre zeroes) led to $E_n \to 0$ at a rate that was faster than the speed with which $\|x-x_n\|_\infty \to 0$. The collocation points $\{t_1,\ldots,t_n\}$ are said to be points of superconvergence with respect to the solution x_n over $[a,b]$.

3.3 Nyström methods

A central feature of the error analysis for degenerate kernel and projection methods is the justifiable assumption that $\|\mathcal{K}-\mathcal{K}_n\| \to 0$ as $n\to\infty$, where $\mathcal{K}_n$ denotes the associated approximation of the integral operator $\mathcal{K}$. With degenerate kernel methods, $\mathcal{K}_n$ is a degenerate kernel integral operator; and for projection methods, $\mathcal{K}_n = \mathcal{P}_n\mathcal{K}$. As discussed above, this leads to a straightforward error analysis based on the geometric series theorem.

In contrast, quadrature-based discretizations satisfy the relation

$$\|\mathcal{K}-\mathcal{K}_n\| \geq \|\mathcal{K}\|, \qquad n\geq 1$$

As a consequence, the convergence analysis for the Nyström method must be something different than that used for degenerate kernel methods and projection methods.

The first convergence analysis for the Nyström method, to this author's knowledge, was given by Kantorovich and Krylov [38, p. 103]. Their analysis is complicated, but it is complete and is equivalent to the bounds of some later authors. Its significance appears to have been overlooked by later researchers. The 1948 paper of Kantorovich [37] contains a general schema for analyzing discretizations of operator equations, and using it he gives another convergence analysis for the solution at the node points. Yet another early analysis is given by Bückner [21] using arguments related to those for degenerate kernel methods

and using piecewise constant interpolation to extend the nodal solution to the full interval.

For an approach that leads to the way in which the Nyström method is currently analysed, begin by defining

$$E_n(s,t) = \int_a^b K(s,v)K(v,t)dv - \sum_{j=1}^{n} w_j K(s,t_j)K(t_j,t) \tag{22}$$

With continuous kernel functions K and standard quadrature schemes that are convergent on $C[a,b]$, Mysovskih [45] showed that

$$E_n(s,t) \to 0 \quad \text{as} \quad n \to \infty \tag{23}$$

uniformly in (s,t). He used this to give a more transparent convergence analysis for Nyström's method. The convergence result (23) shows, implicitly, that in the context of $C[a,b]$, we have that

$$\|(\mathcal{K} - \mathcal{K}_n)\mathcal{K}\|, \ \|(\mathcal{K} - \mathcal{K}_n)\mathcal{K}_n\| \to 0 \quad \text{as} \quad n \to \infty \tag{24}$$

This follows easily from the formulas

$$\begin{aligned} \|(\mathcal{K} - \mathcal{K}_n)\mathcal{K}\| &= \max_{a \le s \le b} \int_a^b |E_n(s,t)|\, dt \\ \|(\mathcal{K} - \mathcal{K}_n)\mathcal{K}_n\| &= \max_{a \le s \le b} \sum_{j=1}^{n} |w_j E_n(s,t_j)| \end{aligned} \tag{25}$$

Anselone and Moore [3] were interested in freeing the error analysis from the specific form of the integral equation (1) and its approximation (9). They found that such an argument using $E_n(s,t)$ in (25) could be avoided within an operator theoretic framework that was based on the following three assumptions.

A1. $\mathcal{K}$ and $\mathcal{K}_n$, $n \ge 1$, are bounded linear operators from a Banach space $\mathcal{X}$ into itself.

A2. $\mathcal{K}_n x \to \mathcal{K}x$ as $n \to \infty$ for all $x \in \mathcal{X}$.

A3. $\{\mathcal{K}_n x : n \ge 1 \text{ and } \|x\| \le 1\}$ has compact closure in $\mathcal{X}$.

From these hypotheses, it follows that $\|(\mathcal{K} - \mathcal{K}_n)\mathcal{K}\|$ and $\|(\mathcal{K} - \mathcal{K}_n)\mathcal{K}_n\|$ converge to zero as $n \to \infty$. To prove this, begin by letting B denote the set in **A3**; its closure $\overline{B}$ is compact. Then

$$\begin{aligned} \|(\mathcal{K} - \mathcal{K}_n)\mathcal{K}_n\| &= \sup_{\|x\| \le 1} \|(\mathcal{K} - \mathcal{K}_n)\mathcal{K}_n x\| \\ &\le \sup_{z \in \overline{B}} \|(\mathcal{K} - \mathcal{K}_n)z\| \end{aligned}$$

In addition, **A3** implies the family $\{\mathcal{K}_n\}$ is uniformly bounded; and thus it is an equicontinuous family on any bounded subset of $\mathcal{X}$. It is straightforward to

show that pointwise convergence of a sequence of functions on a compact set is uniform. Combining these results leads to the convergence $\|(\mathcal{K}-\mathcal{K}_n)\,\mathcal{K}_n\| \to 0$.

An approximating family $\{\mathcal{K}_n\}$ satisfying **A1-A3** is said to be 'pointwise convergent and collectively compact'. This framework turns out to be quite important as there are important extensions of the standard Nyström approximation (9) for which one cannot show directly, as in (25), that $\|(\mathcal{K}-\mathcal{K}_n)\,\mathcal{K}\|$ and $\|(\mathcal{K}-\mathcal{K}_n)\,\mathcal{K}_n\|$ converge to zero. Product quadrature methods for treating singular kernel functions are examples. In addition, the family of approximating operators $\{\mathcal{K}_n\}$ for both degenerate kernel and projection methods satisfy **A3**.

In the error analysis, the earlier argument (13) is replaced by the following:

$$\frac{1}{\lambda}\left\{I+(\lambda I-\mathcal{K})^{-1}\mathcal{K}_n\right\}(\lambda I-\mathcal{K}_n) = I+\frac{1}{\lambda}(\lambda I-\mathcal{K})^{-1}(\mathcal{K}-\mathcal{K}_n)\mathcal{K}_n \tag{26}$$

This identity originates from the following.

$$\begin{aligned}(\lambda I-\mathcal{S})^{-1} &= \frac{1}{\lambda}\left[I+(\lambda I-\mathcal{S})^{-1}\,\mathcal{S}\right]\\ &\approx \frac{1}{\lambda}\left[I+(\lambda I-\mathcal{T})^{-1}\,\mathcal{S}\right]\end{aligned}$$

$$\frac{1}{\lambda}\left[I+(\lambda I-\mathcal{T})^{-1}\,\mathcal{S}\right](\lambda I-\mathcal{S}) = I+\frac{1}{\lambda}\,(\lambda I-\mathcal{T})^{-1}\,(\mathcal{T}-\mathcal{S})\,\mathcal{S}$$

We assume $(\lambda I-\mathcal{K})^{-1}$ exists. From (26) and using $\|(\mathcal{K}-\mathcal{K}_n)\,\mathcal{K}_n\| \to 0$, we can show that $(\lambda I-\mathcal{K}_n)^{-1}$ exists and is uniformly bounded for all sufficiently large n. We can solve (26) for $(\lambda I-\mathcal{K}_n)^{-1}$ to obtain bounds on it that are uniform for sufficiently large n. Letting $(\lambda I-\mathcal{K})\,x=y$ and $(\lambda I-\mathcal{K}_n)\,x_n=y$, we have

$$x-x_n = (\lambda I-\mathcal{K}_n)^{-1}\,(\mathcal{K}x-\mathcal{K}_n x) \tag{27}$$

This shows that the speed of convergence of x_n to x is at least as rapid as the speed of convergence of $\mathcal{K}_n x$ to $\mathcal{K}x$. A more complete discussion of collectively compact operator approximation theory is given in [2].

Other researchers developed related ideas, and the best known are probably those of Stummel [58] (involving the concept of *discrete compactness*) and Vainikko [60] (involving the concept of *compact approximation*). Their frameworks are more general in that the approximating equations can be defined on separate Banach spaces $\mathcal{X}_n$, $n \geq 1$. Another approach to understanding the Nyström method was given by Noble [47]. He developed an alternative framework using the language of prolongation and restriction operators, in some ways reminiscent of the original work of Kantorovich [37], but simpler and with new insights.

Using the abstract framework of collectively compact operators, a number of extensions of the Nyström method have been analyzed. We discuss two of them.

3.3.1 Product integration

Consider singular kernel functions such as $K(s,t) = \log|s-t|$ or $|s-t|^{-\alpha}$, $\alpha < 1$. These define compact integral operators on $C[a,b]$; but an approximation of the associated integral operator based on standard numerical integration is a poor idea. To introduce the main idea of product integration, consider the particular kernel function

$$K(s,t) = L(s,t)\log|s-t|$$

with $L(s,t)$ a well-behaved kernel function.

As a particular case of product integration, let

$$a = t_0 < t_1 < \cdots < t_n = b$$

Let $[z(t)]_n$ denote the piecewise linear interpolant to $z(t)$ with node points $\{t_0, \ldots, t_n\}$. Define the approximation of $\mathcal{K}x$ by

$$\mathcal{K}x(s) \approx \int_a^b [L(s,t)x(t)]_n \log|s-t|\, dt \equiv \mathcal{K}_n x(s)$$

It is straightforward to show that

$$\mathcal{K}_n x(s) = \sum_{j=0}^{n} w_j(s) L(s,t_j)\, x(t_j) \tag{28}$$

This is often called the product trapezoidal approximation.

The approximating equation $(\lambda I - \mathcal{K}_n)\, x_n = y$ can be dealt with in exactly the same manner as in (8)-(11) for the original Nyström method. These ideas for product integration were introduced in [4], [6], motivated in part by Young [66].

For the error analysis, the family $\{\mathcal{K}_n\}$ can be shown to be a pointwise convergent and collectively compact family. The error analysis reduces to that already done for such families, with bounds on the speed of convergence obtainable from (27). It should be noted that it was not possible to show directly for (28) the required convergence in (24).

Much research has been done on product integration methods. De Hoog and Weiss [24] give asymptotic error estimates when $x(t)$ is a smooth function. Among the more important results are those showing that the solution x of such equations $(\lambda I - \mathcal{K})\, x = y$ are usually poorly behaved around the endpoints of $[a,b]$; cf. Graham [27], Richter [52], and Schneider [53]. The latter paper [53] also discusses how to grade the mesh $\{t_i\}$ so as to compensate for the bad behaviour in the solution around the endpoints. For a general discussion of product integration, see [11, §4.2].

3.3.2 The eigenvalue problem

Consider finding the eigenvalues λ and eigenfunctions $x \neq 0$ for the compact integral operator $\mathcal{K}$,

$$\int_a^b K(s,t)x(t)\,dt = \lambda x(s), \qquad a \leq s \leq b, \quad \lambda \neq 0 \tag{29}$$

This is a very old problem, and there is a large research literature on both it and its numerical solution. For a bibliography and some discussion of the early literature, see Bückner [21]. Among the papers in the research literature on the numerical solution of the problem, I particularly note [5], [8], Brakhage [18], Bramble and Osborn [19], Vainikko [59], and Wielandt [64]. The book of Chatelin [23, Chap. 4] gives a general presentation which includes a number of these results, and it contains an up-to-date bibliography of the field.

Let $\lambda_0 \neq 0$ be an eigenvalue of $\mathcal{K}$. Let $\varepsilon > 0$ be chosen so that the set $\mathcal{F} \equiv \{\lambda : |\lambda - \lambda_0| \leq \varepsilon\}$ contains no other eigenvalues of $\mathcal{K}$ and also does not contain 0. Let $\{\mathcal{K}_n\}$ be a collectively compact and pointwise convergent family of approximations to $\mathcal{K}$ on a Banach space $\mathcal{X}$ Let

$$\sigma_n = \left\{\lambda_1^{(n)}, \ldots, \lambda_{r_n}^{(n)}\right\}$$

denote the eigenvalues of $\mathcal{K}_n$ that are located in $\mathcal{F}$. It can be shown that for n sufficiently large, σ_n is contained in the interior of $\mathcal{F}$.

Define

$$E(\lambda_0) = \frac{1}{2\pi i}\int_{|\mu-\lambda_0|=e} (\mu I - \mathcal{K})^{-1}\,d\mu$$

$E(\lambda_0)$ is the *spectral projection* associated with λ_0. $E(\lambda_0)\mathcal{X}$ is the finite-dimensional subspace of simple and generalized eigenvectors associated with the eigenvalue λ_0 for $\mathcal{K}$,

$$E(\lambda_0)\mathcal{X} = \text{Null}\left((\lambda_0 I - \mathcal{K})^{\nu(\lambda_0)}\right)$$

with $\nu(\lambda_0)$ the index of the eigenvalue λ_0.

Define

$$E_n(\sigma_n) = \frac{1}{2\pi i}\int_{|\mu-\lambda_0|=\varepsilon} (\mu I - \mathcal{K}_n)^{-1}\,d\mu$$

$E_n(\sigma_n)\mathcal{X}$ is the direct sum of the subspaces of the simple and generalized eigenvectors associated with the eigenvalues of $\mathcal{K}_n$ contained in the approximating set σ_n,

$$E_n(\sigma_n)\mathcal{X} = \text{Null}\left(\left(\lambda_1^{(n)} I - \mathcal{K}\right)^{\nu\left(\lambda_1^{(n)}\right)}\right) \oplus \cdots \oplus \text{Null}\left(\left(\lambda_r^{(n)} I - \mathcal{K}\right)^{\nu\left(\lambda_r^{(n)}\right)}\right)$$

It is shown in [5] that the approximating eigenvalues in σ_n converge to λ_0,

$$\max_{\lambda\in\sigma_n} |\lambda - \lambda_0| \to 0 \quad \text{as} \quad n \to \infty$$

Also, for every simple and generalized eigenvector $\varphi \in E(\lambda_0)\mathcal{X}$,

$$E_n(\sigma_n)\varphi \to \varphi \quad \text{as} \quad n \to \infty$$

The element $E_n(\sigma_n)\varphi$ is a sum of simple and generalized eigenvectors associated with approximating eigenvalues in σ_n. Error bounds are also given in [5], [8]; and Bramble and Osborn [19] give both bounds and a beautifully simple way to improve the convergence in the case of an eigenvalue with multiplicity greater than 1. These results also apply to degenerate kernel methods and projection methods since the associated approximations $\mathcal{K}_n$ can be shown to be collectively compact and pointwise convergent.

Related and independent error analysis results are given by Vainikko [59], and they are for a more general discretization framework than that of Anselone and Moore.

3.4 Iterative variants

There are iterative variants of all of the numerical methods discussed above. The linear systems for all of these numerical methods result in dense linear systems, say of order n, and then the cost of solution is $O(n^3)$. In addition, with both degenerate kernel methods and projection methods, the elements of the coefficient matrix are integrals which are usually evaluated numerically. With the collocation method these coefficients are single integrals, and with Galerkin's method, they are double integrals. The cost of evaluating the coefficient matrix is generally $O(n^2)$, although the constant of proportionality may be quite large. Evaluating the coefficient matrix for a Nyström method is also $O(n^2)$, but now each coefficient is only a single evaluation of the kernel function.

Most standard iteration methods for solving linear systems of order n, including Krylov subspace methods, lead to a cost of $O(n^2)$, which is consistent with the cost of setting up the coefficient matrix. Two-grid methods were introduced by Brakhage [17] and then developed much more extensively in [7], [9] for both linear and nonlinear problems. These methods also have a cost of $O(n^2)$. In [30] Hackbusch developed a fast multigrid iteration method with a cost of $O(n)$ for solving these linear systems; but the cost of setting up the linear system is still $O(n^2)$. For a much more extensive look at iterative variants of the methods of this paper, see [11, Chap. 6].

4 Boundary integral equation methods

A major use of integral equations has been to reformulate problems for partial differential equations (PDEs), and historically this dates back over 150 years. For example, the classical Neumann series (circa 1873) for solving integral equations refers to work of Carl Neumann in which he was considering a problem in potential theory. For a more complete look at this historical background, see Bateman [15] and Lonseth [44].

Along with the growth from the 1950's onward of finite element methods for solving elliptic PDEs, there was also interest in developing 'boundary element methods' (BEM) for solving 'boundary integral equation' (BIE) reformulations of elliptic PDEs. These integral equation reformulations reduce by 1 the dimensionality of the boundary value problem; and sometimes the solution of interest is needed only on the boundary of the region on which the original PDE is defined. The engineering literature on BEM is enormous, and there are several annual meetings on various aspects of the topic. In the community of researchers devoted to the numerical solution of Fredholm integral equations, the numerical solution of BIE has been a major focus from the late 1970's to the present day.

There are a number of ways to approach the development of BIE reformulations and their numerical analysis, and my perspective is biased by my own work in the area. Among introductions, I refer the reader to the books of [11, Chaps. 7-9], [32], [36], [41], [49], and [63]; for planar BIE problems, see the extensive survey article of Sloan [55]. A survey of the numerical solution of BIE for Laplace's equation is given in [10].

There are many outstanding research papers, and I can only refer to a few of them here; see [11] for a much more extensive bibliography. In his 1968 paper [62], Wendland laid the foundation for collocation methods for solving BIE reformulations in $\mathbb{R}^3$. In [31], Hackbusch and Nowak gave a fast way to set up and solve discretizations of BIE, with a total cost of $O\left(n \log^d n\right)$, d a small integer. An alternative fast method of solution, the fast multipole method, is given by Greengard and Rokhlin [28].

A true extension of the finite element method from PDE to BIE, retaining its variational framework, was introduced by Stephan and Wendland in [57]. This included BIE that were integral equations of the first kind ($\lambda = 0$ in (1)) as well as of the second kind. This framework opened a very fruitful approach to solving BIE, and it is still a very active area of research. An extended discussion of this finite element method is given in [63], and there is a large research literature on it.

Additional comments. Although a number of papers are given in the following bibliography, a much more complete list is given in [11], and additional discussions of the literature are given at the conclusions of the various chapters therein.

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