CLASSIFICATION OF SECOND-ORDER PDES

Recall: the algebraic equation

$$A\,x^2 + B\,xy + C\,y^2 + D\,x + E\,y + F = 0$$

represents a (possibly degenerate) conic section. Denote the discriminant $\Delta = B^2 - 4\,A\,C$. Then the equation represents

an ellipse, if $\Delta < 0$

a parabola, if $\Delta = 0$

a hyperbola, if $\Delta > 0$

Consider a second order partial differential equation of two independent variables

$$A\,u_{xx} + B\,u_{xy} + C\,u_{yy} = F(x, y, u, u_x, u_y)$$

with constants A, B and C, and function $F(x, y, u, p, q)$.

For the PDE, again define the discriminant $\Delta = B^2 - 4\,A\,C$. Then the PDE is

elliptic, if $\Delta < 0$

parabolic, if $\Delta = 0$

hyperbolic, if $\Delta > 0$

For parabolic and hyperbolic PDEs, one of the independent variables, say y, will be replaced by t and can be usually interpreted as a time variable.

REPRESENTATIVE SECOND-ORDER PDES

We will discuss some finite difference schemes for solving the following representative second-order PDEs.

Poisson equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y), \quad (x,y) \in \Omega \subset \mathbb{R}^2$$

where Ω is an open, bounded, connected planar region. This equation is elliptic.

The important special case with $f(x,y) = 0$ is called the Laplace equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad (x,y) \in \Omega \subset \mathbb{R}^2$$

For both equations, need additional conditions on the boundary $\partial\Omega$ to uniquely determine a solution.

Heat equation:

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + f(x,t), \quad x \in (0,L),\ t > 0$$

where, a and L are positive constants.

To determine a unique solution, need additional conditions on the boundary defined by $x = 0$ and $x = L$, and an initial condition

$$u(x,0) = u_0(x), \quad x \in [0,L]$$

Wave equation:

$$\frac{\partial^2 u}{\partial t^2} = a\frac{\partial^2 u}{\partial x^2} + f(x,t), \quad x \in (0,L),\ t > 0$$

Again, a and L are positive constants.

To determine a unique solution, need additional conditions on the boundary defined by $x = 0$ and $x = L$, and *two* initial conditions

$$u(x,0) = u_0(x),\ u_t(x,0) = v_0(x), \quad x \in [0,L]$$

Two initial conditions are needed since the PDE is second-order in t.

POISSON EQUATION

We will focus on the boundary value problem for the Poisson equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y), \quad (x, y) \in \Omega$$

together with the Dirichlet boundary condition

$$u(x, y) = g(x, y), \quad (x, y) \in \partial\Omega$$

Boundary conditions involving first order partial derivatives of the unknown function are also possible.

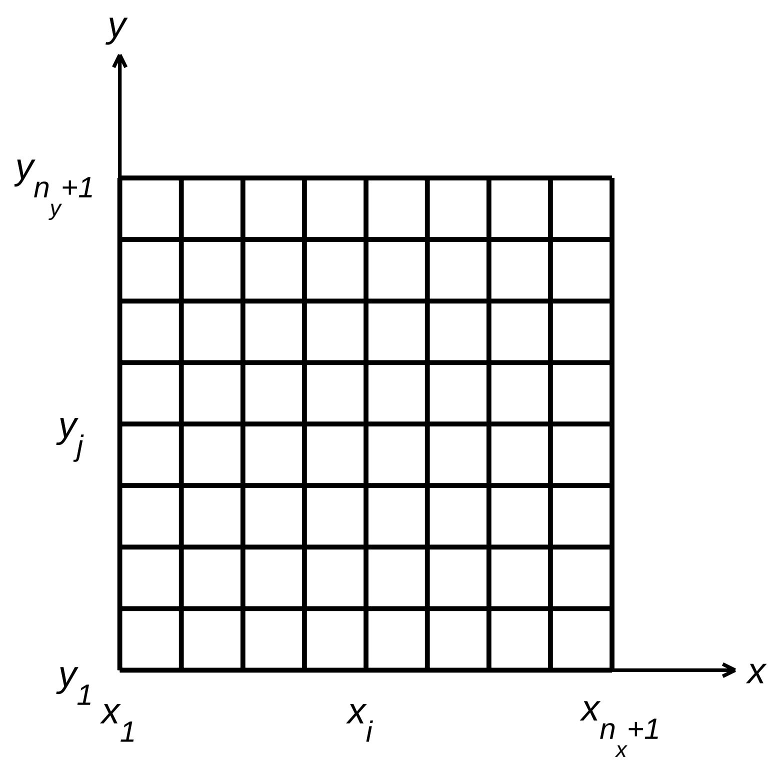
In the following, we derive a finite difference scheme for solving the boundary value problem. The derivation is done for the case of a square: $\Omega = (0, 1) \times (0, 1)$. The idea of the derivation can be applied to more general Ω. Note that the boundary $\partial\Omega$ consists of four line segments, which are the four sides of the square.

Finite Difference Grid

We first introduce a finite difference grid for $\overline{\Omega} = [0,1] \times [0,1]$. Divide the x interval $[0,1]$ into n_x equal parts and denote $h_x = 1/n_x$ the x stepsize. Similarly, we divide the y interval $[0,1]$ into n_y equal parts and denote $h_y = 1/n_y$ the y stepsize. Then the grid points are

$$(x_i, y_j), \quad 1 \le i \le n_x + 1,\ 1 \le j \le n_y + 1$$

where $x_i = (i-1)\,h_x,\ y_j = (j-1)\,h_y$.



Five Point Scheme

Consider the differential equation at an interior grid point (x_i, y_j), $2 \le i \le n_x$, $2 \le j \le n_y$. Use the three-point central difference to approximate the second derivative:

$$\frac{\partial^2 u}{\partial x^2}(x_i, y_j) \approx \frac{u(x_{i+1}, y_j) - 2\,u(x_i, y_j) + u(x_{i-1}, y_j)}{h_x^2}$$

$$\frac{\partial^2 u}{\partial x^2}(x_i, y_j) \approx \frac{u(x_i, y_{j+1}) - 2\,u(x_i, y_j) + u(x_i, y_{j-1})}{h_y^2}$$

Denote $f_{ij} = f(x_i, y_j)$, and by u_{ij} the finite difference approximation of $u(x_i, y_j)$. Then we have the following difference equations at the interior grid points:

$$\frac{u_{i+1,j} - 2\,u_{i,j} + u_{i-1,j}}{h_x^2} + \frac{u_{i,j+1} - 2\,u_{i,j} + u_{i,j-1}}{h_y^2} = f_{ij}, \quad 2 \le i \le n_x,\ 2 \le j \le n_y \tag{1}$$

These equations are supplemented by the discrete Dirichlet boundary conditions:

$$u_{ij} = g_{ij}, \quad i = 1 \text{ or } n_x + 1, \text{ or } j = 1 \text{ or } n_y + 1 \tag{2}$$

We can then use the boundary conditions (2) in the difference equations (1) to obtain a linear system for the unknowns u_{ij} for $2 \le i \le n_x$, $2 \le j \le n_y$.

It can be shown that the method is second-order accurate:

$$\max_{\substack{1\le i\le n_x+1\\ 1\le j\le n_y+1}} |u(x_i, y_j) - u_{ij}| = O(h_x^2 + h_y^2)$$

assuming the solution $u(x, y)$ has several continuous partial derivatives.

Implementation

The resulting linear system from the five point scheme can be solved by direct methods (Gaussian elimination method and its variants), as well as iterative methods. The form of the five point scheme is natural for the application of an iterative method.

For simplicity in writing, consider the particular case $n_x = n_y = n$, $h_x = h_y = h$. Then the difference equation (1) can be rewritten as

$$u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4\,u_{i,j} = h^2 f_{ij}$$

$$2 \leq i, j \leq n$$

or

$$u_{i,j} = \frac{1}{4}\,(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1}) - \frac{h^2}{4}\,f_{ij}$$

$$2 \leq i, j \leq n$$

An application of Gauss-Seidel iteration method is: given an initial guess $\{u_{i,j}^{(1)}\}_{2\le i,j\le n}$, for $k = 1, 2, \cdots$, determine $\{u_{i,j}^{(k+1)}\}_{2\le i,j\le n}$ recursively by

for $i = 2, \ldots, n$

for $j = 2, \ldots, n$

$$u_{i,j}^{(k+1)} = \frac{1}{4}(u_{i+1,j}^{(k)} + u_{i-1,j}^{(k+1)} + u_{i,j+1}^{(k)}$$

$$+u_{i,j-1}^{(k+1)}) - \frac{h^2}{4} f_{ij}$$

end

end

When i or j is 1 or $n+1$, the boundary condition is used: $u_{i,j}^{(k+1)} = g_{i,j}$.

EXAMPLE

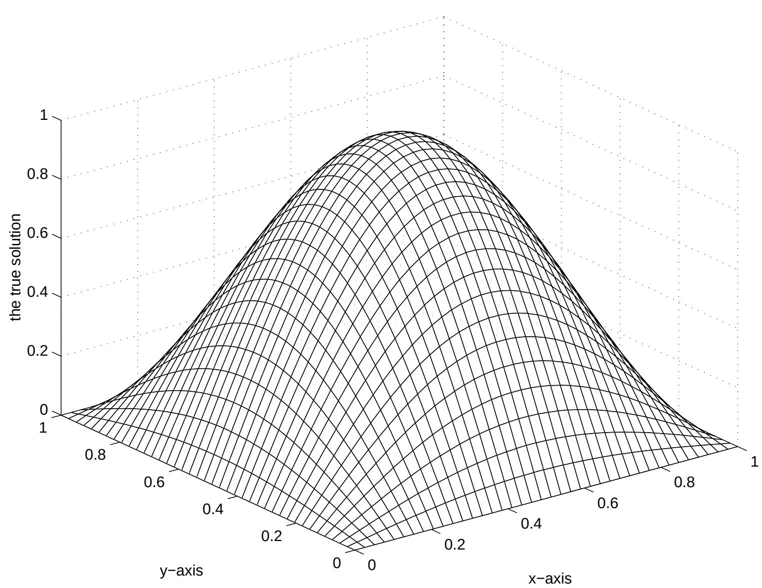
Consider the boundary value problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = -2\,\pi^2 \sin(\pi x)\,\sin(\pi y) \\ \qquad\qquad 0 < x, y < 1, \\ u(x,y) = 0, \quad x = 0,1 \text{ or } y = 0,1. \end{cases}$$

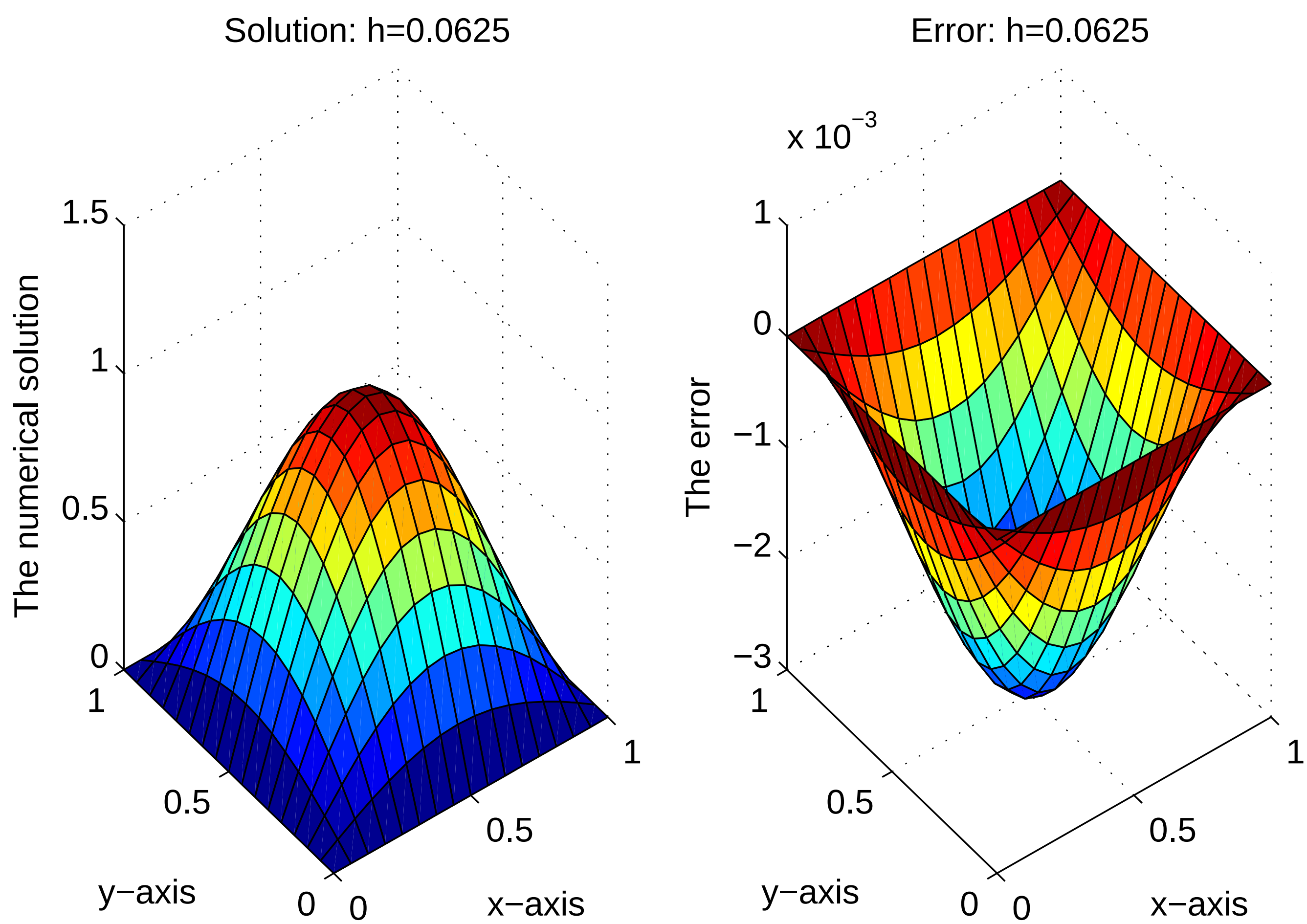
The true solution is

$$u(x,y) = \sin(\pi x)\,\sin(\pi y)$$

and it is shown in the figure below.



We use the five point scheme to solve the problem. The numerical results with $n = 16$ are given in the figure.



Maximum errors $\max_{1\leq i,j\leq n+1} |u(x_i, y_j) - u_{ij}|$ for various n are listed in the following table.

n	Max Error	Ratio
4	5.3029E − 2	
8	1.2951E − 2	4.09
16	3.2190E − 3	4.02
32	8.0347E − 4	4.01

Notice that as the value n is doubled, i.e. the grid size h is halved, the maximum error is reduced by a factor of approximately 4. This confirms the theoretical error bound of order two.

To see more precisely the error behavior, we calculate the solution errors at six selective node points, and the corresponding ratios. The six points are $P_1 = (1/4, 1/4)$, $P_2 = (1/2, 1/4)$, $P_3 = (3/4, 1/4)$, $P_4 = (1/4, 1/2)$, $P_5 = (1/2, 1/2)$, and $P_6 = (1/4, 3/4)$.

	$n = 4$	$n = 8$	R	$n = 16$	R
P_1	-2.65E-2	-6.48E-3	4.09	-1.61E-3	4.02
P_2	-3.75E-2	-9.16E-3	4.09	-2.28E-3	4.02
P_3	-2.65E-2	-6.48E-3	4.09	-1.61E-3	4.02
P_4	-3.75E-2	-9.16E-3	4.09	-2.28E-3	4.02
P_5	-5.30E-2	-1.30E-2	4.09	-3.22E-3	4.02
P_6	-2.65E-2	-6.48E-3	4.09	-1.61E-3	4.02

Again, notice that the ratios are all close to 4.

It is possible to do extrapolation to improve the efficiency. Under certain smoothness assumptions on the solution, the following asymptotic error expansion can be proved:

$$u(x_i, y_j) - u_{ij} = h^2 D(x_i, y_j) + O(h^4) \qquad (3)$$

for any grid point (x_i, y_j). Here $D(x, y)$ denotes some function determined from a boundary value problem involving u in the data.

Denoting $u_h(x, y)$ for the numerical solution at a grid point (x, y) with the grid size h, we rewrite (3) as

$$u(x_i, y_j) - u_h(x_i, y_j) = h^2 D(x_i, y_j) + O(h^4) \quad (4)$$

If (x, y) is a grid point corresponding to the grid size $2h$, then it is a grid point also with the grid size h. From (4) we have

$$\begin{aligned} u(x,y) - u_h(x,y) &= h^2 D(x,y) + O(h^4) \\ u(x,y) - u_{2h}(x,y) &= (2h)^2 D(x,y) + O(h^4) \end{aligned}$$

Eliminating the term $D(x, y)$ from the two relations, we obtain

$$u(x,y) - \tilde{u}_h(x,y) = O(h^4)$$

where $\tilde{u}_h$ is the extrapolated solution defined by

$$\tilde{u}_h(x,y) = \frac{4\,u_h(x,y) - u_{2h}(x,y)}{3}$$

i.e., without much additional effort, we obtain a fourth-order method.

Extrapolated solution errors for the boundary value problem being solved are given in the next table.

	$n = 8$	$n = 16$	R
P_1	2.04E−4	1.25E−5	16.35
P_2	2.89E−4	1.77E−5	16.35
P_3	2.04E−4	1.25E−5	16.35
P_4	2.89E−4	1.77E−5	16.35
P_5	4.09E−4	2.50E−5	16.35
P_6	2.04E−4	1.25E−5	16.35

Let us compare the accuracy of the extrapolated solution with the five point difference solution. Take $n = 16$ as an example. We notice that, at the six selected points, the errors of the extrapolated solution are around 2×10^{-5}, whereas that of the difference solution are around -2×10^{-3}. In other words, here with extrapolation, for comparable amount of calculations, the accuracy of the numerical solution is increased about 100 times.