GENERAL ERROR FORMULA

In general,

$$y_{n+1} = y_n + h\, f(x_n, y_n), \quad n = 0, 1, ..., N-1$$

$$\begin{aligned} Y(x_{n+1}) &= Y(x_n) + h\, Y'(x_n) + \frac{h^2}{2} Y''(\xi_n) \\ &= Y(x_n) + h\, f(x_n, Y(x_n)) + \frac{h^2}{2} Y''(\xi_n) \end{aligned}$$

with some $x_n \le \xi_n \le x_{n+1}$.

We will use this as the starting point of our discussion of the error in Euler's method. In particular,

$$\begin{aligned} Y(x_{n+1}) - y_{n+1} = \; & Y(x_n) - y_n \\ & + h\, [\, f(x_n, Y(x_n)) - f(x_n, y_n)] \\ & + \frac{h^2}{2} Y''(\xi_n) \end{aligned}$$

ERROR ANALYSIS - SPECIAL CASES

We begin with a couple of special cases, to obtain some additional intuition on the behaviour of the error. Consider

$$y' = 2x, \quad y(0) = 0$$

This has the solution $Y(x) = x^2$. Euler's method becomes

$$y_{n+1} = y_n + 2x_n h, \quad y_0 = 0$$

$$\begin{aligned} y_1 &= y_0 + 2x_0 h &&= x_1 x_0 \\ y_2 &= y_1 + 2x_1 h &&= x_1 x_0 + 2x_1 h = x_2 x_1 \\ y_3 &= y_2 + 2x_2 h &&= x_2 x_1 + 2x_2 h = x_3 x_2 \end{aligned}$$

By an induction argument,

$$y_n = x_n x_{n-1}, \quad n \geq 1$$

For the error,

$$Y(x_n) - y_n = x_n^2 - x_n x_{n-1} = x_n h$$

Note that $Y(x_n) - y_n \propto h$ at each fixed x_n.

Return to our error equation

$$\begin{aligned} Y(x_{n+1}) - y_{n+1} &= Y(x_n) - y_n \\ &\quad + h\,[\,f(x_n, Y(x_n)) - f(x_n, y_n)] \\ &\quad + \frac{h^2}{2} Y''(\xi_n) \end{aligned} \tag{1}$$

With the mean value theorem,

$$f(x_n, Y(x_n)) - f(x_n, y_n) = \frac{\partial f(x_n, \zeta_n)}{\partial y}\,[Y(x_n) - y_n]$$

for some ζ_n between $Y(x_n)$ and y_n. As shorthand, use $e_h(x) = Y(x) - y_h(x)$. Then we can write

$$e_h(x_{n+1}) = \left[1 + h\frac{\partial f(x_n, \zeta_n)}{\partial y}\right] e_h(x_n) + \frac{h^2}{2} Y''(\xi_n) \tag{2}$$

with $e_h(x_0) = 0$. We also will assume henceforth that

$$K \equiv \max_{\substack{x_0 \le x \le b \\ -\infty < y < \infty}} \left| \frac{\partial f(x, y)}{\partial y} \right| < \infty$$

Consider those differential equations with

$$\frac{\partial f(x,y)}{\partial y} \leq 0, \quad x_0 \leq x \leq b, \quad -\infty < y < \infty$$

Then

$$-1 \leq 1 + h\frac{\partial f(x_n, \zeta_n)}{\partial y} \leq 1$$

provided h is chosen sufficiently small, *e.g.* if

$$h < \frac{1}{K}$$

Using this in our error formula (2),

$$|e_h(x_{n+1})| \leq |e_h(x_n)| + \frac{h^2}{2}\left\|Y''\right\|_\infty, \quad n \geq 0 \quad (3)$$

in which

$$\left\|Y''\right\|_\infty = \max_{x_0 \leq t \leq b}\left|Y''(t)\right|$$

Using induction with (3), we can prove

$$|e_h(x_n)| \leq \frac{h}{2}(x_n - x_0)\left\|Y''\right\|_\infty$$

Again the error is bounded by something of the form $c(x_n)h$.

EXAMPLE

Solve the initial value problems

$$y' = -y + \cos x - \sin x, \quad 0 \le x \le 5, \quad y(0) = 2$$

Its true solution is

$$Y(x) = e^{-x} + \cos x$$

Then

$$Y''(x) = e^{-x} - \cos x$$

$$\|Y''\|_\infty = \max_{0 \le x \le 5} |Y''(x)| \doteq 1.0442$$

We solve the problem with Euler's method on $[0, 5]$.

For this differential equation,

$$f(x, y) = -y + \cos x - \sin x, \qquad \frac{\partial f}{\partial y} = -1$$

Then the error bound becomes

$$\begin{aligned} |e_h(x_n)| &\le \frac{h}{2}(x_n - x_0) \|Y''\|_\infty \\ &= 0.5221 h x_n \end{aligned}$$

The following table gives the error when solving this problem with $h = .1$ on the interval $[0, 5]$. The final column is the error bound, and it clearly exceeds the magnitude of the actual error at each value of x_n.

x_n	y_n	$Y(x_n) - y_n$	$0.5221hx_n$
0	2.00000000	0.000000	0.00000
1	.91533345	−.007152	.05221
2	−.28535798	.004546	.10442
3	−.97071646	.030511	.15663
4	−.67539538	.040067	.20884
5	.27163709	.018763	.26105

GENERAL ERROR ANALYSIS

Return to

$$e_h(x_{n+1}) = \left[1 + h\frac{\partial f(x_n, \zeta_n)}{\partial y}\right] e_h(x_n) + \frac{h^2}{2} Y''(\xi_n)$$

in which

$$e_h(x) = Y(x) - y_h(x)$$

As before, we assume

$$K \equiv \max_{\substack{-\infty < y < \infty \\ x_0 \le x \le b}} \left|\frac{\partial f(x, y)}{\partial y}\right| < \infty$$

This is much more restrictive than is needed, but it simplifies the statement and analysis of the error in solving differential equations. From this we can show

$$|e_h(x_n)| \le e^{(x_n - x_0)K} |e_h(x_0)| + h\frac{e^{(x_n - x_0)K} - 1}{2K} \left\|Y''\right\|_\infty$$

for all points $x_0 \le x_n \le b$. In theory, we would have $e_h(x_0) = 0$. But we can also examine what would happen if we allowed for errors in the choice of y_0.

EXAMPLE

Recall the last example, of solving

$$y' = -y + \cos x - \sin x, \quad 0 \le x \le 5, \quad y(0) = 2$$

whose true solution is $Y(x) = e^{-x} + \cos x$. Again,

$$f(x, y) = -y + \cos x - \sin x, \qquad \frac{\partial f}{\partial y} = -1$$

and then $K = 1$. Recall

$$Y''(x) = e^{-x} - \cos x$$
$$\|Y''\|_\infty = \max_{0 \le x \le 5} |Y''(x)| \doteq 1.0442$$

and also assume $e_h(0) = 0$. Then our general error bound becomes

$$\begin{aligned} |e_h(x_n)| &\le h\frac{e^{(x_n - x_0)K} - 1}{2K} \|Y''\|_\infty \\ &= h\frac{e^{x_n} - 1}{2}(1.0442) \end{aligned}$$

This is consistent with our earlier bound since

$$e^{x_n} - 1 \ge x_n$$

ASYMPTOTIC ERROR FORMULA

We see from the earlier results that the error satisfies

$$|Y(x_n) - y_n| \leq c\,(x_n)\,h, \quad x_0 \leq x_n \leq b$$

for some number $c\,(x_n)$. We can improve upon this. Namely, it can be shown that

$$Y(x_n) - y_h(x_n) = D(x_n)h + O\left(h^2\right)$$

The term $O(h^2)$ denotes a quantity which is bounded by a constant times h^2. We say the term being bounded is of <u>order</u> h^2. Thus

$$Y(x_n) - y_h(x_n) \approx D(x_n)h \tag{4}$$

for smaller values of h. The function $D(x)$ satisfies a particular differential equation, but it can seldom be found in practice since it depends on the solution $Y(x)$. Instead we use (4) to justify using Richardson extrapolation to estimate the error.

RICHARDSON EXTRAPOLATION

Consider solving the initial value problem

$$y' = f(x, y), \quad x_0 \le x \le b, \qquad y(x_0) = Y_0$$

with Euler's method, as suppose we do it twice, using stepsizes of h and $2h$. Denote the respective numerical solutions by $y_h(x_n)$ and $y_{2h}(x_n)$. Then from (4) above, and at a generic node point x,

$$\begin{aligned} Y(x) - y_h(x) &\approx D(x)h \\ Y(x) - y_{2h}(x) &\approx D(x)\,(2h) \end{aligned}$$

Multiply the first equation by 2 and then subtract the second equation. This yields

$$Y(x) - 2y_h(x) + y_{2h}(x) \approx 0$$
$$Y(x) \approx 2y_h(x) - y_{2h}(x)$$

This last formula is called "Richardson's extrapolation formula" for Euler's method. We can also use it to estimate the error.

$$\begin{aligned} Y(x) - y_h(x) &\approx [2y_h(x) - y_{2h}(x)] - y_h(x) \\ &= y_h(x) - y_{2h}(x) \end{aligned}$$

The formula

$$Y(x) - y_h(x) \approx y_h(x) - y_{2h}(x)$$

is called "Richardson's error estimation formula" for Euler's method.

EXAMPLE: Recall the last example, of solving

$$y' = -y + \cos x - \sin x, \quad 0 \le x \le 5, \quad y(0) = 2$$

whose true solution is $Y(x) = e^{-x} + \cos x$. We do so with stepsizes $h = .1$ and $2h = .2$.

x_n	$y_h(x_n)$	$Y(x_n) - y_h(x_n)$	$y_h(x) - y_{2h}(x)$
0	2.00000000	0.000000	0.00000
1	.91533345	$-.007152$	-0.007593
2	$-.28535798$	.004546	0.004420
3	$-.97071646$	.030511	0.031741
4	$-.67539538$	.040067	0.041539
5	.27163709	.018763	0.018820