DIFFERENTIAL EQUATIONS

A principal model of physical phenomena.

The equation:

$$y' = f(x, y)$$

The initial value:

$$y(x_0) = Y_0$$

Find solution $Y(x)$ on some interval $x_0 \leq x \leq b$. Together these two conditions constitute an *initial value problem*.

We will study methods for solving systems of first order equations, but we begin with a single equation.

Many of the crucial ideas in the numerical analysis arise from properties of the original equation.

SPECIAL CASES

1. $y'(x) = \lambda y(x) + b(x),\ x \geq x_0$;
   $f(x, z) = \lambda z + b(x)$.
   General solution:

   $$Y(x) = ce^{\lambda x} + \int_{x_0}^{x} e^{\lambda(x-t)} b(t)dt$$

   with c arbitrary. With $y(x_0) = Y_0$,

   $$Y(x) = Y_0 e^{\lambda(x-x_0)} + \int_{x_0}^{x} e^{\lambda(x-t)} b(t)dt$$

2. $y'(x) = ay\,(x)^2$; $f(x, z) = az^2$.
   General solution:

   $$Y(x) = \frac{-1}{ax+c}, \quad c \text{ arbitrary}$$

   With $y(x_0) = Y_0$, use

   $$c = -ax_0 - \frac{1}{Y_0}$$

3. $y'(x) = -[y(x)]^2 + y(x)$; $f(x,z) = -z^2 + z$.
   General solution:

$$Y(x) = \frac{1}{1 + ce^{-x}}$$

4. "Separable equations": $y'(x) = g(y(x))h(x)$;
   $f(x,z) = g(z)h(x)$.
   General solution: Write

$$\frac{1}{g(y)}\frac{dy}{dx} = h(x)$$

   Let $z = y(x)$, $dz = y'(x)dx$. Evaluate the integrals in

$$\int \frac{dz}{g(z)} = \int h(x)dx$$

   Replace z by $Y(x)$ and solve for $Y(x)$, if possible.

DIRECTION FIELDS

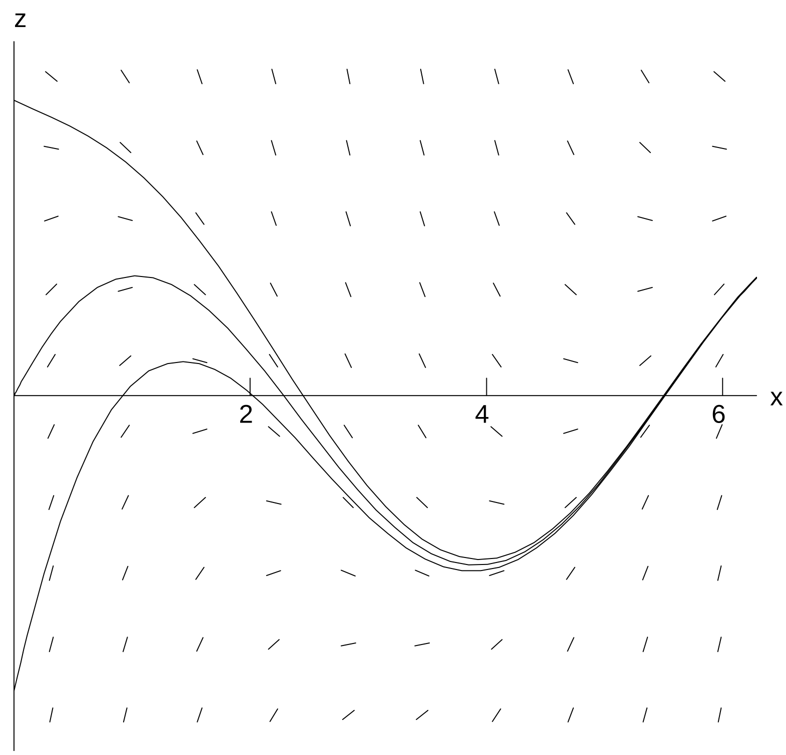
At each point (x, y) at which the function f is defined, evaluate it to get $f(x, y)$. Then draw in a small line segment at this point with slope $f(x, y)$. With enough of these, we have a picture of how the solutions behave for the differential equation

$$y' = f(x, y)$$

Consider the differential equation

$$y' = -y + 2\cos x$$

We can draw direction fields by hand by the method described above, by using the Matlab program given in the book; or we can use the Matlab program provided in the class account.



Direction field for $y' = -y + 2\cos x$. Also shown are example solution curves

SOLVABILITY THEORY

Consider whether there is a function $Y(x)$ which satisfies

$$y' = f(x, y), \quad x \geq x_0, \quad y(x_0) = Y_0 \qquad (1)$$

Assume there is some *open* set D that is subset of the xy-plane and that contains (x_0, Y_0), for which:

1. If two points (x, y) and (x, z) are contained in D, then the line segment joining them is also contained in D.

2. $f(x, y)$ is continuous for all points (x, y) contained in D.

3. $\partial f(x, y)/\partial y$ is continuous for all points (x, y) contained in D.

Then there is an interval $[c, d]$ containing x_0 and there is a unique function $Y(x)$ defined on $[c, d]$ which satisfies (1), with the graph of $Y(x)$ contained in D.

THE LIPSCHITZ CONDITION

The preceding condition on the partial derivative of f is an easy way to specify that the following condition is satisfied. It is the condition that is really needed.
The Lipschitz condition: There is a non-negative constant K for which

$$|f(x,y) - f(x,z)| \le K\,|y-z|$$

for all points (x,y), (x,z) in the region D. In practice, we use

$$K = \max_{(x,y)\in D} \left|\frac{\partial f(x,y)}{\partial y}\right|$$

The Lipschitz condition occurs throughout our treatment of both the theory of differential equations and the theory of the numerical methods for their solution.

For this course, we simplify matters by assuming

$$K = \max_{\substack{-\infty<y<\infty \\ x_0\le x\le b}} \left|\frac{\partial f(x,y)}{\partial y}\right| < \infty$$

with $[x_0, b]$ the interval on which we are solving the initial value problem.

EXAMPLE

Let $\alpha > 0$ be a given constant, and consider solving

$$y' = \frac{2x}{\alpha^2} y^2, \quad x \ge 0, \quad y(0) = 1$$

Then the partial derivative is

$$f_y(x, y) = \frac{4xy}{\alpha^2}$$

and $f_y(0, 1) = 0$. Thus $f_y(x, y)$ is small for (x, y) near to $(0, 1)$, and it is continuous for all (x, y). Choose

$$D = \{(x, y) : |x| \le 1, |y| \le B\}$$

for some $B > 0$. Then there is a solution $Y(x)$ on some interval $[c, d]$ containing $x_0 = 0$. How big is $[c, d]$? In this case,

$$Y(x) = \frac{\alpha^2}{\alpha^2 - x^2}, \quad -\alpha < x < \alpha$$

If α is small, then the interval is small.

IMPROVED SOLVABILITY THEORY

Assume there is a Lipschitz constant K for which f satisfies

$$|f(x,y) - f(x,z)| \le K\,|y - z|$$

for all (x,y), (x,z) satisfying

$$x_0 \le x \le b, \quad -\infty < y, z < \infty$$

Then the initial value problem

$$y' = f(x,y), \quad x_0 \le x_0 \le b, \quad y(x_0) = Y_0$$

has a solution $Y(x)$ on the entire interval $[x_0, b]$.

<u>Example</u>: Consider $y' = y + g(x)$ with $g(x)$ continuous for all x. Then

$$y' = y + g(x), \quad y(x_0) = Y_0$$

has a solution $Y(x)$ has a unique continuous solution for $-\infty < x < \infty$.

STABILITY

The concept of *stability* refers in a loose sense to what happens to the solution $Y(x)$ of an initial value problem if we make a small change in the *data*, which includes both the differential equation and the initial value.

If small changes in the data lead to large changes in the solution, then we say the initial value problem is *unstable* or *ill-conditioned*; whereas if small changes in the data lead to small changes in the solution, we call the problem *stable* or *well-conditioned*.

EXAMPLE

Consider solving

$$y' = 100y - 101e^{-x}, \quad y(0) = 1 \tag{2}$$

This has a solution of $Y(x) = e^{-x}$.

Now consider the *perturbed problem*

$$y' = 100y - 101e^{-x}, \quad y(0) = 1 + \epsilon$$

where ϵ is some small number. The solution of this is $Y_\epsilon(x) = e^{-x} + \epsilon e^{100x}$, and

$$Y_\epsilon(x) - Y(x) = \epsilon e^{100x}$$

Thus $Y_\epsilon(x) - Y(x)$ increases very rapidly as x increases, and we say (2) is an "unstable" or "ill-conditioned" problem.