ITERATION METHODS

These are methods which compute a sequence of progressively accurate iterates to approximate the solution of $Ax = b$.

We need such methods for solving many large linear systems. Sometimes the matrix is too large to be stored in the computer memory, making a direct method too difficult to use.

More importantly, the operations cost of $\frac{2}{3}n^3$ for Gaussian elimination is too large for most large systems. With iteration methods, the cost can often be reduced to something of cost $O\left(n^2\right)$ or less. Even when a special form for A can be used to reduce the cost of elimination, iteration will often be faster.

There are other, more subtle, reasons, which we do not discuss here.

JACOBI'S ITERATION METHOD

We begin with an example. Consider the linear system

$$\begin{array}{rcrcrcl} 9x_1 & + & x_2 & + & x_3 & = & b_1 \\ 2x_1 & + & 10x_2 & + & 3x_3 & = & b_2 \\ 3x_1 & + & 4x_2 & + & 11x_3 & = & b_3 \end{array}$$

In equation $\#k$, solve for x_k:

$$\begin{aligned} x_1 &= \tfrac{1}{9}[b_1 - x_2 - x_3] \\ x_2 &= \tfrac{1}{10}[b_2 - 2x_1 - 3x_3] \\ x_3 &= \tfrac{1}{11}[b_3 - 3x_1 - 4x_2] \end{aligned}$$

Let $x^{(0)} = \left[x_1^{(0)}, x_2^{(0)}, x_3^{(0)}\right]^T$ be an initial guess to the solution x. Then define

$$\begin{aligned} x_1^{(k+1)} &= \tfrac{1}{9}\left[b_1 - x_2^{(k)} - x_3^{(k)}\right] \\ x_2^{(k+1)} &= \tfrac{1}{10}\left[b_2 - 2x_1^{(k)} - 3x_3^{(k)}\right] \\ x_3^{(k+1)} &= \tfrac{1}{11}\left[b_3 - 3x_1^{(k)} - 4x_2^{(k)}\right] \end{aligned}$$

for $k = 0, 1, 2, \ldots$. This is called the *Jacobi iteration method* or the *method of simultaneous replacements*.

NUMERICAL EXAMPLE.

Let $b = [10,\ 19,\ 0]^T$.

The solution is $x = [1,\ 2,\ -1]^T$.

To measure the error, we use

$$Error = \|x - x^{(k)}\| = \max_i \left| x_i - x_i^{(k)} \right|$$

k	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$	*Error*	*Ratio*
0	0	0	0	2.00E + 0	
1	1.1111	1.9000	0	1.00E + 0	0.500
2	0.9000	1.6778	−0.9939	3.22E − 1	0.322
3	1.0351	2.0182	−0.8556	1.44E − 1	0.448
4	0.9819	1.9496	−1.0162	5.06E − 2	0.349
5	1.0074	2.0085	−0.9768	2.32E − 2	0.462
6	0.9965	1.9915	−1.0051	8.45E − 3	0.364
7	1.0015	2.0022	−0.9960	4.03E − 3	0.477
8	0.9993	1.9985	−1.0012	1.51E − 3	0.375
9	1.0003	2.0005	−0.9993	7.40E − 4	0.489
10	0.9999	1.9997	−1.0003	2.83E − 4	0.382
30	1.0000	2.0000	−1.0000	3.01E − 11	0.447
31	1.0000	2.0000	−1.0000	1.35E − 11	0.447

GAUSS-SEIDEL ITERATION METHOD

Again consider the linear system

$$\begin{array}{rcrcrcl} 9x_1 & + & x_2 & + & x_3 & = & b_1 \\ 2x_1 & + & 10x_2 & + & 3x_3 & = & b_2 \\ 3x_1 & + & 4x_2 & + & 11x_3 & = & b_3 \end{array}$$

and solve for x_k in equation $\#k$:

$$\begin{aligned} x_1 &= \tfrac{1}{9}[b_1 - x_2 - x_3] \\ x_2 &= \tfrac{1}{10}[b_2 - 2x_1 - 3x_3] \\ x_3 &= \tfrac{1}{11}[b_3 - 3x_1 - 4x_2] \end{aligned}$$

Now immediately use every new iterate:

$$\begin{aligned} x_1^{(k+1)} &= \tfrac{1}{9}\left[b_1 - x_2^{(k)} - x_3^{(k)}\right] \\ x_2^{(k+1)} &= \tfrac{1}{10}\left[b_2 - 2x_1^{(k+1)} - 3x_3^{(k)}\right] \\ x_3^{(k+1)} &= \tfrac{1}{11}\left[b_3 - 3x_1^{(k+1)} - 4x_2^{(k+1)}\right] \end{aligned}$$

for $k = 0, 1, 2, \ldots$. This is called the *Gauss-Seidel iteration method* or the *method of successive replacements*.

NUMERICAL EXAMPLE.

Let $b = [10,\ 19,\ 0]^T$.
The solution is $x = [1,\ 2,\ -1]^T$.
To measure the error, we use

$$Error = \|x - x^{(k)}\| = \max_i \left| x_i - x_i^{(k)} \right|$$

k	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$	*Error*	*Ratio*
0	0	0	0	$2.00E+0$	
1	1.1111	1.6778	-0.9131	$3.22E-1$	0.161
2	1.0262	1.9687	-0.9958	$3.13E-2$	0.097
3	1.0030	1.9981	-1.0001	$3.00E-3$	0.096
4	1.0002	2.0000	-1.0001	$2.24E-4$	0.074
5	1.0000	2.0000	-1.0000	$1.65E-5$	0.074
6	1.0000	2.0000	-1.0000	$2.58E-6$	0.155

The values of *Ratio* do not approach a limiting value with larger values of the iteration index k.

A GENERAL SCHEMA

Rewrite $Ax = b$ as

$$Nx = b + Px \qquad (1)$$

with $A = N - P$ a ***splitting*** of A. Choose N to be nonsingular. Usually we want $Nz = f$ to be easily solvable for arbitray f. The iteration method is

$$Nx^{(k+1)} = b + Px^{(k)}, \qquad k = 0, 1, 2, \ldots, \qquad (2)$$

EXAMPLE. Let N be the diagonal of A, and let $P = N - A$. The iteration method is the Jacobi method:

$$a_{i,i}x_i^{(k+1)} = b_i - \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{i,j}x_j^{(k)}, \quad 1 \le i \le n$$

for $k = 0, 1, \ldots$.

EXAMPLE. Let N be the lower triangular part of A, including its diagonal, and let $P = N - A$. The iteration method is the Gauss-Seidel method:

$$\sum_{j=1}^{i} a_{i,j} x_j^{(k+1)} = b_i - \sum_{j=i+1}^{n} a_{i,j} x_j^{(k)}, \quad 1 \le i \le n$$

for $k = 0, 1, \ldots$.

EXAMPLE. Another method could be defined by letting N be the tridiagonal matrix formed from the diagonal, super-diagonal, and sub-diagonal of A, with $P = N - A$:

$$N = \begin{bmatrix} a_{1,1} & a_{1,2} & 0 & \cdots & 0 \\ a_{2,1} & a_{2,2} & a_{2,3} & \ddots & \vdots \\ 0 & & \ddots & & 0 \\ \vdots & \ddots & a_{n-1,n-2} & a_{n-1,n-1} & a_{n-1,n} \\ 0 & \cdots & 0 & a_{n,n-1} & a_{n,n} \end{bmatrix}$$

Solving $Nx^{(k+1)} = b + Px^{(k)}$ uses the algorithm for tridiagonal systems from §6.4.

CONVERGENCE

When does the iteration method (2) converge? Subtract (2) from (1), obtaining

$$\begin{aligned} N\left(x - x^{(k+1)}\right) &= P\left(x - x^{(k)}\right) \\ x - x^{(k+1)} &= N^{-1}P\left(x - x^{(k)}\right) \\ e^{(k+1)} &= Me^{(k)}, \quad M = N^{-1}P \end{aligned} \quad (3)$$

with $e^{(k)} \equiv x - x^{(k)}$

Return now to the matrix and vector norms of §6.5. Then

$$\left\|e^{(k+1)}\right\| \le \|M\| \left\|e^{(k)}\right\|, \qquad k \ge 0$$

Thus the error $e^{(k)}$ converges to zero if $\|M\| < 1$, with

$$\left\|e^{(k)}\right\| \le \|M\|^k \left\|e^{(0)}\right\|, \qquad k \ge 0$$

EXAMPLE. For the earlier example with the Jacobi method,

$$\begin{aligned} x_1^{(k+1)} &= \tfrac{1}{9}\left[b_1 - x_2^{(k)} - x_3^{(k)}\right] \\ x_2^{(k+1)} &= \tfrac{1}{10}\left[b_2 - 2x_1^{(k)} - 3x_3^{(k)}\right] \\ x_3^{(k+1)} &= \tfrac{1}{11}\left[b_3 - 3x_1^{(k)} - 4x_2^{(k)}\right] \end{aligned}$$

$$M = \begin{bmatrix} 0 & -\frac{1}{9} & -\frac{1}{9} \\ -\frac{2}{10} & 0 & -\frac{3}{10} \\ -\frac{3}{11} & -\frac{4}{11} & 0 \end{bmatrix}$$

$$\|M\| = \frac{7}{11} \doteq 0.636$$

This is consistent with the earlier table of values, although the actual convergence rate was better than predicted by (3).

EXAMPLE. For the earlier example with the Gauss-Seidel method,

$$
\begin{aligned}
x_1^{(k+1)} &= \tfrac{1}{9}\left[b_1 - x_2^{(k)} - x_3^{(k)}\right] \\
x_2^{(k+1)} &= \tfrac{1}{10}\left[b_2 - 2x_1^{(k+1)} - 3x_3^{(k)}\right] \\
x_3^{(k+1)} &= \tfrac{1}{11}\left[b_3 - 3x_1^{(k+1)} - 4x_2^{(k+1)}\right]
\end{aligned}
$$

$$
\begin{aligned}
M &= \begin{bmatrix} 9 & 0 & 0 \\ 2 & 10 & 0 \\ 3 & 4 & 11 \end{bmatrix}^{-1} \begin{bmatrix} 0 & -1 & -1 \\ 0 & 0 & -3 \\ 0 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & -\frac{1}{9} & -\frac{1}{9} \\ 0 & \frac{1}{45} & -\frac{5}{18} \\ 0 & \frac{1}{45} & \frac{13}{99} \end{bmatrix} \\
\|M\| &= 0.3
\end{aligned}
$$

This too is consistent with the earlier numerical results.

DIAGONALLY DOMINANT MATRICES

Matrices A for which

$$\left|a_{i,i}\right| > \sum_{\substack{j=1 \\ j\neq i}}^{n} \left|a_{i,j}\right|, \qquad i = 1, \ldots, n$$

are called *diagonally dominant*. For the Jacobi iteration method,

$$M = \begin{bmatrix} 0 & -\dfrac{a_{1,2}}{a_{1,1}} & \cdots & -\dfrac{a_{1,n}}{a_{1,1}} \\ -\dfrac{a_{2,1}}{a_{2,2}} & 0 & & -\dfrac{a_{2,n}}{a_{2,2}} \\ \vdots & & \ddots & \vdots \\ -\dfrac{a_{n,1}}{a_{n,n}} & \cdots & -\dfrac{a_{n,n-1}}{a_{n,n}} & 0 \end{bmatrix}$$

With diagonally dominant matrices A,

$$\|M\| = \max_{1\leq i\leq n} \sum_{\substack{j=1 \\ j\neq i}}^{n} \left|\frac{a_{i,j}}{a_{i,i}}\right| < 1 \qquad (4)$$

Thus the Jacobi iteration method for solving $Ax = b$ is convergent.

GAUSS-SEIDEL ITERATION

Assuming A is diagonally dominant, we can show that the Gauss-Seidel iteration will also converge. However, constructing $M = N^{-1}P$ is not reasonable for this method and an alternative approach is needed. Return to the error equation

$$Ne^{(k+1)} = Pe^{(k)}$$

and write it in component form for the Gauss-Seidel method:

$$\sum_{j=1}^{i} a_{i,j} e_j^{(k+1)} = -\sum_{j=i+1}^{n} a_{i,j} e_j^{(k)}, \quad 1 \le i \le n$$

$$e_i^{(k+1)} = -\sum_{j=1}^{i-1} \frac{a_{i,j}}{a_{i,i}} e_j^{(k+1)} - \sum_{j=i+1}^{n} \frac{a_{i,j}}{a_{i,i}} e_j^{(k)} \qquad (5)$$

Introduce

$$\alpha_i = \sum_{j=1}^{i-1} \left| \frac{a_{i,j}}{a_{i,i}} \right|, \qquad \beta_i = \sum_{j=i+1}^{n} \left| \frac{a_{i,j}}{a_{i,i}} \right|, \quad 1 \le i \le n$$

with $\alpha_1 = \beta_n = 0$. Taking bounds in (5),

$$\left| e_i^{(k+1)} \right| \le \alpha_i \left\| e^{(k+1)} \right\| + \beta_i \left\| e^{(k)} \right\|, \quad i = 1, ..., n \quad (6)$$

Let ℓ be an index for which

$$\left| e_\ell^{(k+1)} \right| = \max_{1 \le i \le n} \left| e_i^{(k+1)} \right| = \left\| e^{(k+1)} \right\|$$

Then using $i = \ell$ in (6),

$$\left\| e^{(k+1)} \right\| \le \alpha_\ell \left\| e^{(k+1)} \right\| + \beta_\ell \left\| e^{(k)} \right\|$$

$$\left\| e^{(k+1)} \right\| \le \frac{\beta_\ell}{1 - \alpha_\ell} \left\| e^{(k)} \right\|$$

Define

$$\eta = \max_i \frac{\beta_i}{1 - \alpha_i}$$

Then

$$\left\| e^{(k+1)} \right\| \le \eta \left\| e^{(k)} \right\|$$

For A diagonally dominant, it can be shown that

$$\eta \leq \|M\| \tag{7}$$

where $\|M\|$ is for the definition of M for the Jacobi method, given earlier in (4) as

$$\|M\| = \max_{1 \leq i \leq n} \sum_{\substack{j=1 \\ j \neq i}}^{n} \left| \frac{a_{i,j}}{a_{i,i}} \right| = \max_{1 \leq i \leq n} (\alpha_i + \beta_i) < 1$$

Consequently, for A diagonally dominant, the Gauss-Seidel method also converges and it does so more rapidly than the Jacobi method in most cases.

Showing (7) follows by showing

$$\frac{\beta_i}{1 - \alpha_i} - (\alpha_i + \beta_i) \geq 0, \qquad 1 \leq i \leq n$$

For our earlier example with A of order 3, we have $\mu = 0.375$ This is not as good as computing $\|M\|$ directly for the Gauss-Seidel method, but it does show that the rate of convergence is better than for the Jacobi method.

CONVERGENCE: AN ADDENDUM

Since

$$\|M\| = \|N^{-1}P\| \leq \|N^{-1}\| \, \|P\|,$$

$\|M\| < 1$ is satisfied if N satisfies

$$\|N^{-1}\| \, \|P\| < 1$$

Using $P = N - A$, this can be rewritten as

$$\|A - N\| < \frac{1}{\|N^{-1}\|}$$

We also want to choose N so that systems $Nz = f$ is ‘easily solvable’.

GENERAL CONVERGENCE THEOREM:

$$Nx^{(k+1)} = b + Px^{(k)}, \qquad k = 0, 1, 2, \ldots,$$

will converge, for all right sides b and all initial guesses $x^{(0)}$, if and only if all eigenvalues λ of $M = N^{-1}P$ satisfy

$$|\lambda| < 1$$

This is the basis of deriving other splittings $A = N - P$ that lead to convergent iteration methods.

RESIDUAL CORRECTION METHODS

N be an invertible approximation of the matrix A; let $x^{(0)} \approx x$ for the solution of $Ax = b$. Define

$$r^{(0)} = b - Ax^{(0)}$$

Since $Ax = b$ for the true solution x,

$$\begin{aligned} r^{(0)} &= Ax - Ax^{(0)} \\ &= A(x - x^{(0)}) = Ae^{(0)} \end{aligned}$$

with $e^{(0)} = x - x^{(0)}$. Let $\hat{e}^{(0)}$ be the solution of

$$N\hat{e}^{(0)} = r^{(0)}$$

and then define

$$x^{(1)} = x^{(0)} + \hat{e}^{(0)}$$

Repeat this process inductively.

RESIDUAL CORRECTION

For $k = 0, 1, \ldots,$ define

$$\begin{aligned} r^{(k)} &= b - Ax^{(k)} \\ N\hat{e}^{(k)} &= r^{(k)} \\ x^{(k+1)} &= x^{(k)} + \hat{e}^{(k)} \end{aligned}$$

This is the general *residual correction method*.

To see how this fits into our earlier framework, proceed as follows:

$$\begin{aligned} x^{(k+1)} &= x^{(k)} + \hat{e}^{(k)} = x^{(k)} + N^{-1}r^{(k)} \\ &= x^{(k)} + N^{-1}(b - Ax^{(k)}) \end{aligned}$$

Thus,

$$\begin{aligned} Nx^{(k+1)} &= Nx^{(k)} + b - Ax^{(k)} \\ &= b + (N - A)x^{(k)} \\ &= b + Px^{(k)} \end{aligned}$$

Sometimes the residual correction scheme is a preferable way of approaching the development of an iterative method.