MATRICES

Matrices are rectangular arrays of real or complex numbers. With them, we define arithmetic operations that are generalizations of those for real and complex numbers. The general form a matrix of order $m \times n$ is

$$A = \begin{bmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{m,1} & \cdots & a_{m,n} \end{bmatrix}$$

We say it has *order* $m \times n$. Matrices that consist of a single column are called *column vectors*, and those consisting of a single row are called *row vectors*. In both cases, they will have properties identical to the geometric vectors studied earlier in mulitvariable calculus. I assume that most of you have seen this material previously in a course named linear algebra, matrix algebra, or something similar. Section 6.2 in the text is intended as both a quick introduction and review of this material.

MATRIX ADDITION

Let $A = \left[a_{i,j}\right]$ and $B = \left[b_{i,j}\right]$ be matrices of order $m \times n$. Then

$$C = A + B$$

is another matrix of order $m \times n$, with

$$c_{i,j} = a_{i,j} + b_{i,j}$$

EXAMPLE.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ -1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 2 & 5 \\ 6 & 5 \end{bmatrix}$$

MULTIPLICATION BY A CONSTANT

$$c \begin{bmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{m,1} & \cdots & a_{m,n} \end{bmatrix} = \begin{bmatrix} ca_{1,1} & \cdots & ca_{1,n} \\ \vdots & \ddots & \vdots \\ ca_{m,1} & \cdots & ca_{m,n} \end{bmatrix}$$

EXAMPLE.

$$5 \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} 5 & 10 \\ 15 & 20 \\ 25 & 30 \end{bmatrix}$$

$$(-1) \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}$$

THE ZERO MATRIX 0

Define the *zero matrix* of order $m \times n$ as the matrix of that order having all zero entries. It is sometimes written as $0_{m\times n}$, but more commonly as simply 0. Then for any matrix A of order $m \times n$,

$$A + 0 = 0 + A = A$$

The zero matrix $0_{m\times n}$ acts in the same role as does the number zero when doing arithmetic with real and complex numbers.

EXAMPLE.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

We denote by $-A$ the solution of the equation

$$A + B = 0$$

It is the matrix obtained by taking the negative of all of the entries in A. For example,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \quad -\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix} = (-1)\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$-\begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} = \begin{bmatrix} -a_{1,1} & -a_{1,2} \\ -a_{2,1} & -a_{2,2} \end{bmatrix}$$

MATRIX MULTIPLICATION

Let $A = \left[a_{i,j}\right]$ have order $m \times n$ and $B = \left[b_{i,j}\right]$ have order $n \times p$. Then

$$C = AB$$

is a matrix of order $m \times p$ and

$$\begin{aligned} c_{i,j} &= A_{i,*}B_{*,j} \\ &= a_{i,1}b_{1,j} + a_{i,2}b_{2,j} + \cdots + a_{i,n}b_{n,j} \end{aligned}$$

or equivalently

$$\begin{aligned} c_{i,j} &= \left[a_{i,1}\ a_{i,2}\ \cdots\ a_{i,n}\right] \begin{bmatrix} b_{1,j} \\ b_{2,j} \\ \vdots \\ b_{n,j} \end{bmatrix} \\ &= a_{i,1}b_{1,j} + a_{i,2}b_{2,j} + \cdots + a_{i,n}b_{n,j} \end{aligned}$$

EXAMPLES

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} 22 & 28 \\ 49 & 64 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 9 & 12 & 15 \\ 19 & 26 & 33 \\ 29 & 40 & 51 \end{bmatrix}$$

$$\begin{bmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{1,1}x_1 + \cdots + a_{1,n}x_n \\ \vdots \\ a_{n,1}x_1 + \cdots + a_{n,n}x_n \end{bmatrix}$$

Thus we write the linear system

$$\begin{aligned} a_{1,1}x_1 + \cdots + a_{1,n}x_n &= b_1 \\ &\vdots \\ a_{n,1}x_1 + \cdots + a_{n,n}x_n &= b_n \end{aligned}$$

as

$$Ax = b$$

THE IDENTITY MATRIX I

For a given integer $n \geq 1$, Define I_n to be the matrix of order $n \times n$ with 1's in all diagonal positions and zeros elsewhere:

$$I_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \dots & & 1 \end{bmatrix}$$

More commonly it is denoted by simply I.

Let A be a matrix of order $m \times n$. Then

$$AI_n = A, \qquad I_m A = A$$

The identity matrix I acts in the same role as does the number 1 when doing arithmetic with real and complex numbers.

THE MATRIX INVERSE

Let A be a matrix of order $n \times n$ for some $n \geq 1$. We say a matrix B is an *inverse* for A if

$$AB = BA = I$$

It can be shown that if an inverse exists for A, then it is unique.

EXAMPLES. If $ad - bc \neq 0$, then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & 1 \\ 1 & -\frac{1}{2} \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix}^{-1} = \begin{bmatrix} 9 & -36 & 30 \\ -36 & 192 & -180 \\ 30 & -180 & 180 \end{bmatrix}$$

Recall the earlier theorem on the solution of linear systems $Ax = b$ with A a square matrix.

Theorem. The following are equivalent statements.

1. For each b, there is exactly one solution x.

2. For each b, there is a solution x.

3. The homogeneous system $Ax = 0$ has only the solution $x = 0$.

4. $\det(A) \neq 0$.

5. A^{-1} exists.

EXAMPLE

$$\det \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = 0$$

Therefore, the linear system

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

is not always solvable, the coefficient matrix does not have an inverse, and the homogeneous system $Ax = 0$ has a solution other than the zero vector, namely

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The arithmetic properties of matrix addition and multiplication are listed on page 256, and some of them require some work to show. For example, consider showing the distributive law for matrix multiplication,

$$(AB)\,C = A\,(BC)$$

with A, B, C matrices of respective orders $m \times n, n \times p$, and $p \times q$, respectively. Writing this out, we want to show

$$\sum_{k=1}^{p} (AB)_{i,k}\, C_{k,l} = \sum_{j=1}^{n} A_{i,j}\, (BC)_{j,l}$$

for $1 \leq i \leq m$, $1 \leq l \leq q$.

With new situations, we often use notation to suggest what should be true. But this is done only after deciding what actually is true. You should read carefully the properties given in the text on page 256.

PARTITIONED MATRICES

Matrices can be built up from smaller matrices; or conversely, we can decompose a large matrix into a matrix of smaller matrices. For example, consider

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 1 \\ 0 & -1 & 5 \end{bmatrix} = \begin{bmatrix} B & c \\ d & e \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \quad c = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad d = \begin{bmatrix} 0 & -1 \end{bmatrix} \quad e = 5$$

Matlab allows you to build up larger matrices out of smaller matrices in exactly this manner; and smaller matrices can be defined as portions of larger matrices. We will often write an $n \times n$ square matrix in terms of its columns:

$$A = \left[A_{*,1}, ..., A_{*,n} \right]$$

For the $n \times n$ identity matrix I, we write

$$I = [e_1, ..., e_n]$$

with e_j denoting a column vector with a 1 in position j and zeros elsewhere.

ARITHMETIC OF PARTITIONED MATRICES

As with matrices, we can do addition and multiplication with partitioned matrices provided the individual constituent parts have the proper orders.

For example, let A, B, C, D be $n \times n$ matrices. Then

$$\begin{bmatrix} I & A \\ B & I \end{bmatrix} \begin{bmatrix} I & C \\ D & I \end{bmatrix} = \begin{bmatrix} I + AD & C + A \\ B + D & I + BC \end{bmatrix}$$

Let A be $n \times n$ and x be a column vector of length n. Then

$$Ax = \left[A_{*,1}, \ldots, A_{*,n}\right] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1 A_{*,1} + \cdots + x_n A_{*,n}$$

Compare this to

$$\begin{bmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{1,1}x_1 + \cdots + a_{1,n}x_n \\ \vdots \\ a_{n,1}x_1 + \cdots + a_{n,n}x_n \end{bmatrix}$$

PARTITIONED MATRICES IN Matlab

In Matlab, matrices can be constructed using smaller matrices. For example, let

$A = [1,\quad 2;\quad 3,\quad 4];\quad x = [5,\quad 6];\quad y = [7,\quad 8]';$

Then

$$B = [A,\quad y;\quad x,\quad 9];$$

forms the matrix

$$B = \begin{bmatrix} 1 & 2 & 7 \\ 3 & 4 & 8 \\ 5 & 6 & 9 \end{bmatrix}$$

ELEMENTARY ROW OPERATIONS

As preparation for the discussion of Gaussian Elimination in Section 6.3, we introduce three elementary row operations on general rectangular matrices. They are:

i) *Interchange of two rows.*

ii) *Multiplication of a row by a nonzero scalar*.

iii) *Addition of a nonzero multiple of one row to another row.*

Consider the rectangular matrix

$$A = \begin{bmatrix} 3 & 3 & 3 & 1 \\ 2 & 2 & 3 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix}$$

We add row 2 times (-1) to row 1, and then add row 3 times (-1) to row 2 to obtain the matrix:

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 2 & 3 & 1 \end{bmatrix}$$

Add row 2 times (-1) to row 1, and to row 3 as well:

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 3 & 1 \end{bmatrix}$$

Add row 1 times (-2) to row 3:

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 \end{bmatrix}$$

Interchange row 1 and row 2:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 1 \end{bmatrix}$$

Finally, we multiply row 3 by $1/3$:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1/3 \end{bmatrix}$$

This is obtained from A using elementary row operations. A reverse sequence of operations of the same type converts this result back to A.

OPERATIONS COUNT

It is important to compare algorithms by comparing both their accuracy and their cost. For cost, we look at the number of arithmetic operations. As a simple example, look at the cost of evaluating

$$b = Ax$$

where A is a square matrix of order $n \times n$ and x is a column vector of length n. Then

$$b_i = \sum_{j=1}^{n} a_{i,j} x_j, \qquad i = 1, \ldots, n.$$

Each component b_i requires n multiplications and $n-1$ additions. Thus the computations of Ax requires n^2 multiplications and $n(n-1) \approx n^2$ additions.

Doubling n increases the cost by a factor of 4.

If A and B are square matrices of order $n \times n$, then calculating AB requires n^3 multiplications and $n^2 (n-1) \approx n^3$ additions. Doubling n increases the cost by a factor of 8.