ERROR IN LINEAR INTERPOLATION

Let $P_1(x)$ denote the linear polynomial interpolating $f(x)$ at x_0 and x_1, with $f(x)$ a given function (e.g. $f(x) = \cos x$). What is the error $f(x) - P_1(x)$?

Let $f(x)$ be twice continuously differentiable on an interval $[a, b]$ which contains the points $\{x_0, x_1\}$. Then for $a \le x \le b$,

$$f(x) - P_1(x) = \frac{(x - x_0)(x - x_1)}{2} f''(c_x)$$

for some c_x between the minimum and maximum of x_0, x_1, and x.

If x_1 and x are 'close to x_0', then

$$f(x) - P_1(x) \approx \frac{(x - x_0)(x - x_1)}{2} f''(x_0)$$

Thus the error acts like a quadratic polynomial, with zeros at x_0 and x_1.

EXAMPLE

Let $f(x) = \log_{10} x$; and in line with typical tables of $\log_{10} x$, we take $1 \leq x, x_0, x_1 \leq 10$. For definiteness, let $x_0 < x_1$ with $h = x_1 - x_0$. Then

$$f''(x) = -\frac{\log_{10} e}{x^2}$$

$$\begin{aligned}
\log_{10} x - P_1(x) &= \frac{(x - x_0)(x - x_1)}{2}\left[-\frac{\log_{10} e}{c_x^2}\right] \\
&= (x - x_0)(x_1 - x)\left[\frac{\log_{10} e}{2c_x^2}\right]
\end{aligned}$$

We usually are interpolating with $x_0 \leq x \leq x_1$; and in that case, we have

$$(x - x_0)(x_1 - x) \geq 0, \qquad x_0 \leq c_x \leq x_1$$

$$(x - x_0)(x_1 - x) \geq 0, \qquad x_0 \leq c_x \leq x_1$$

and therefore

$$\begin{aligned}(x - x_0)(x_1 - x)\left[\frac{\log_{10} e}{2x_1^2}\right] &\leq \log_{10} x - P_1(x) \\ &\leq (x - x_0)(x_1 - x)\left[\frac{\log_{10} e}{2x_0^2}\right]\end{aligned}$$

For $h = x_1 - x_0$ small, we have for $x_0 \leq x \leq x_1$

$$\log_{10} x - P_1(x) \approx (x - x_0)(x_1 - x)\left[\frac{\log_{10} e}{2x_0^2}\right]$$

Typical high school algebra textbooks contain tables of $\log_{10} x$ with a spacing of $h = .01$. What is the error in this case? To look at this, we use

$$0 \leq \log_{10} x - P_1(x) \leq (x - x_0)(x_1 - x)\left[\frac{\log_{10} e}{2x_0^2}\right]$$

By simple geometry or calculus,

$$\max_{x_0 \le x \le x_1} (x - x_0)(x_1 - x) \le \frac{h^2}{4}$$

Therefore,

$$0 \le \log_{10} x - P_1(x) \le \frac{h^2}{4}\left[\frac{\log_{10} e}{2x_0^2}\right] \doteq .0543\frac{h^2}{x_0^2}$$

If we want a *uniform bound* for all points $1 \le x_0 \le 10$, we have

$$0 \le \log_{10} x - P_1(x) \le \frac{h^2 \log_{10} e}{8} \doteq .0543h^2$$

$$0 \le \log_{10} x - P_1(x) \le .0543h^2$$

For $h = .01$, as is typical of the high school text book tables of $\log_{10} x$,

$$0 \le \log_{10} x - P_1(x) \le 5.43 \times 10^{-6}$$

If you look at most tables, a typical entry is given to only four decimal places to the right of the decimal point, e.g.

$$\log 5.41 \doteq .7332$$

Therefore the entries are in error by as much as .00005. Comparing this with the interpolation error, we see the latter is less important than the rounding errors in the table entries.

From the bound

$$0 \le \log_{10} x - P_1(x) \le \frac{h^2 \log_{10} e}{8x_0^2} \doteq .0543\frac{h^2}{x_0^2}$$

we see the error decreases as x_0 increases, and it is about 100 times smaller for points near 10 than for points near 1.

AN ERROR FORMULA: THE GENERAL CASE

Recall the general interpolation problem: find a polynomial $P_n(x)$ for which $\deg(P_n) \le n$

$$P_n(x_i) = f(x_i), \qquad i = 0, 1, \cdots, n$$

with distinct node points $\{x_0, ..., x_n\}$ and a given function $f(x)$. Let $[a, b]$ be a given interval on which $f(x)$ is $(n+1)$-times continuously differentiable; and assume the points $x_0, ..., x_n$, and x are contained in $[a, b]$. Then

$$f(x) - P_n(x) = \frac{(x - x_0)(x - x_1) \cdots (x - x_n)}{(n+1)!} f^{(n+1)}(c_x)$$

with c_x some point between the minimum and maximum of the points in $\{x, x_0, ..., x_n\}$.

$$f(x) - P_n(x) = \frac{(x - x_0)(x - x_1) \cdots (x - x_n)}{(n + 1)!} f^{(n+1)}(c_x)$$

As shorthand, introduce

$$\Psi_n(x) = (x - x_0)(x - x_1) \cdots (x - x_n)$$

a polynomial of degree $n + 1$ with roots $\{x_0, ..., x_n\}$. Then

$$f(x) - P_n(x) = \frac{\Psi_n(x)}{(n + 1)!} f^{(n+1)}(c_x)$$

THE QUADRATIC CASE

For $n = 2$, we have

$$f(x) - P_2(x) = \frac{(x - x_0)(x - x_1)(x - x_2)}{3!} f^{(3)}(c_x) \qquad (*)$$

with c_x some point between the minimum and maximum of the points in $\{x, x_0, x_1, x_2\}$.

To illustrate the use of this formula, consider the case of evenly spaced nodes:

$$x_1 = x_0 + h, \qquad x_2 = x_1 + h$$

Further suppose we have $x_0 \leq x \leq x_2$, as we would usually have when interpolating in a table of given function values (e.g. $\log_{10} x$). The quantity

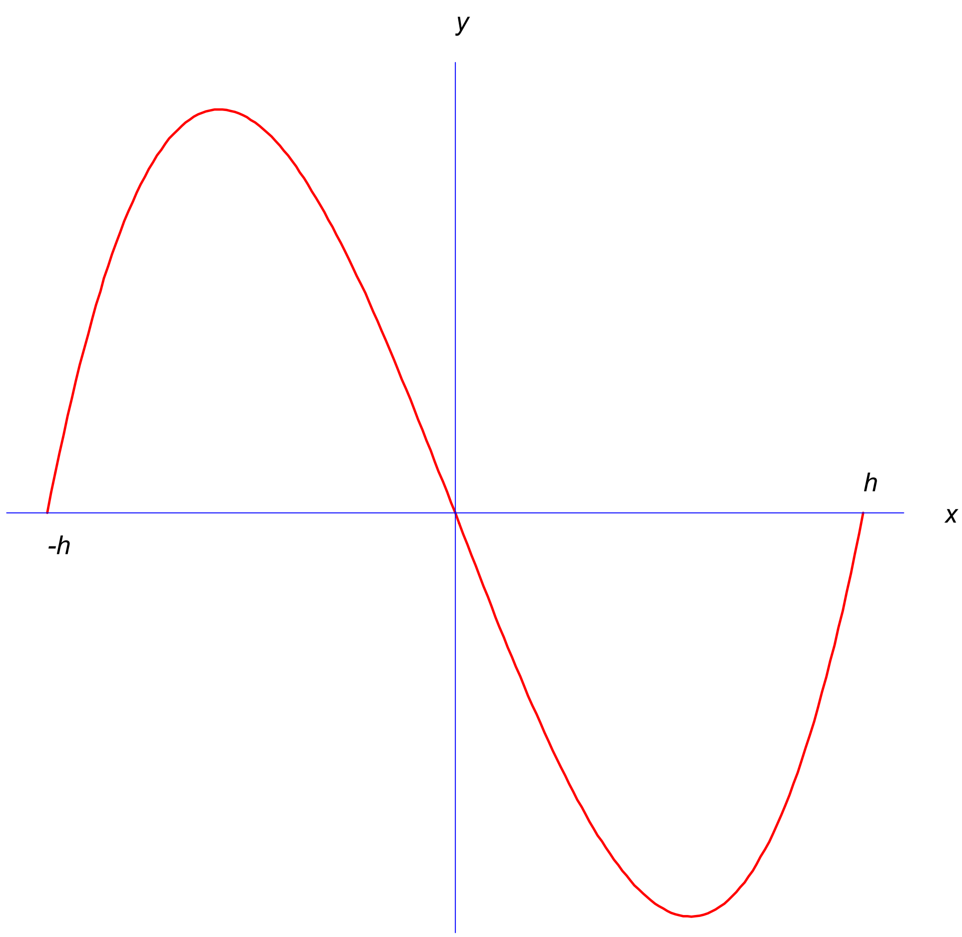
$$\Psi_2(x) = (x - x_0)(x - x_1)(x - x_2)$$

can be evaluated directly for a particular x.

Graph of

$$\Psi_2(x) = (x+h)\,x\,(x-h)$$

using $(x_0, x_1, x_2) = (-h, 0, h)$:



In the formula $(*)$, however, we do not know c_x, and therefore we replace $\left|f^{(3)}(c_x)\right|$ with a maximum of $\left|f^{(3)}(x)\right|$ as x varies over $x_0 \le x \le x_2$. This yields

$$|f(x) - P_2(x)| \le \frac{|\Psi_2(x)|}{3!} \max_{x_0 \le x \le x_2} \left|f^{(3)}(x)\right| \qquad (**)$$

If we want a uniform bound for $x_0 \le x \le x_2$, we must compute

$$\max_{x_0 \le x \le x_2} |\Psi_2(x)| = \max_{x_0 \le x \le x_2} |(x - x_0)(x - x_1)(x - x_2)|$$

Using calculus,

$$\max_{x_0 \le x \le x_2} |\Psi_2(x)| = \frac{2h^3}{3\ \mathrm{sqrt}(3)}, \qquad \text{at } x = x_1 \pm \frac{h}{\mathrm{sqrt}(3)}$$

Combined with $(**)$, this yields

$$|f(x) - P_2(x)| \le \frac{h^3}{9\ \mathrm{sqrt}(3)} \max_{x_0 \le x \le x_2} \left|f^{(3)}(x)\right|$$

for $x_0 \le x \le x_2$.

For $f(x) = \log_{10} x$, with $1 \leq x_0 \leq x \leq x_2 \leq 10$, this leads to

$$\begin{aligned} |\log_{10} x - P_2(x)| &\leq \frac{h^3}{9 \text{ sqrt}(3)} \cdot \max_{x_0 \leq x \leq x_2} \frac{2 \log_{10} e}{x^3} \\ &= \frac{.05572\, h^3}{x_0^3} \end{aligned}$$

For the case of $h = .01$, we have

$$|\log_{10} x - P_2(x)| \leq \frac{5.57 \times 10^{-8}}{x_0^3} \leq 5.57 \times 10^{-8}$$

Question: How much larger could we make h so that quadratic interpolation would have an error comparable to that of linear interpolation of $\log_{10} x$ with $h = .01$? The error bound for the linear interpolation was 5.43×10^{-6}, and therefore we want the same to be true of quadratic interpolation. Using a simpler bound, we want to find h so that

$$|\log_{10} x - P_2(x)| \leq .05572\, h^3 \leq 5 \times 10^{-6}$$

This is true if $h = .04477$. Therefore a spacing of $h = .04$ would be sufficient. A table with this spacing and quadratic interpolation would have an error comparable to a table with $h = .01$ and linear interpolation.

For the case of general n,

$$\begin{aligned} f(x) - P_n(x) &= \frac{(x - x_0) \cdots (x - x_n)}{(n+1)!} f^{(n+1)}(c_x) \\ &= \frac{\Psi_n(x)}{(n+1)!} f^{(n+1)}(c_x) \\ \Psi_n(x) &= (x - x_0)(x - x_1) \cdots (x - x_n) \end{aligned}$$

with c_x some point between the minimum and maximum of the points in $\{x, x_0, ..., x_n\}$. When bounding the error we replace $f^{(n+1)}(c_x)$ with its maximum over the interval containing $\{x, x_0, ..., x_n\}$, as we have illustrated earlier in the linear and quadratic cases.

Consider now the function

$$\frac{\Psi_n(x)}{(n+1)!}$$

over the interval determined by the minimum and maximum of the points in $\{x, x_0, ..., x_n\}$. For evenly spaced node points on $[0, 1]$, with $x_0 = 0$ and $x_n = 1$, we give graphs for $n = 2, 3, 4, 5$ and for $n = 6, 7, 8, 9$ on accompanying pages.

DISCUSSION OF ERROR

Consider the error

$$\begin{aligned} f(x) - P_n(x) &= \frac{(x - x_0) \cdots (x - x_n)}{(n+1)!} f^{(n+1)}(c_x) \\ &= \frac{\Psi_n(x)}{(n+1)!} f^{(n+1)}(c_x) \\ \Psi_n(x) &= (x - x_0)(x - x_1) \cdots (x - x_n) \end{aligned}$$

as n increases and as x varies. As noted previously, we cannot do much with $f^{(n+1)}(c_x)$ except to replace it with a maximum value of $\left| f^{(n+1)}(x) \right|$ over a suitable interval. Thus we concentrate on understanding the size of

$$\frac{\Psi_n(x)}{(n+1)!}$$

ERROR FOR EVENLY SPACED NODES

We consider first the case in which the node points are evenly spaced, as this seems the 'natural' way to define the points at which interpolation is carried out. Moreover, using evenly spaced nodes is the case to consider for table interpolation. What can we learn from the given graphs?

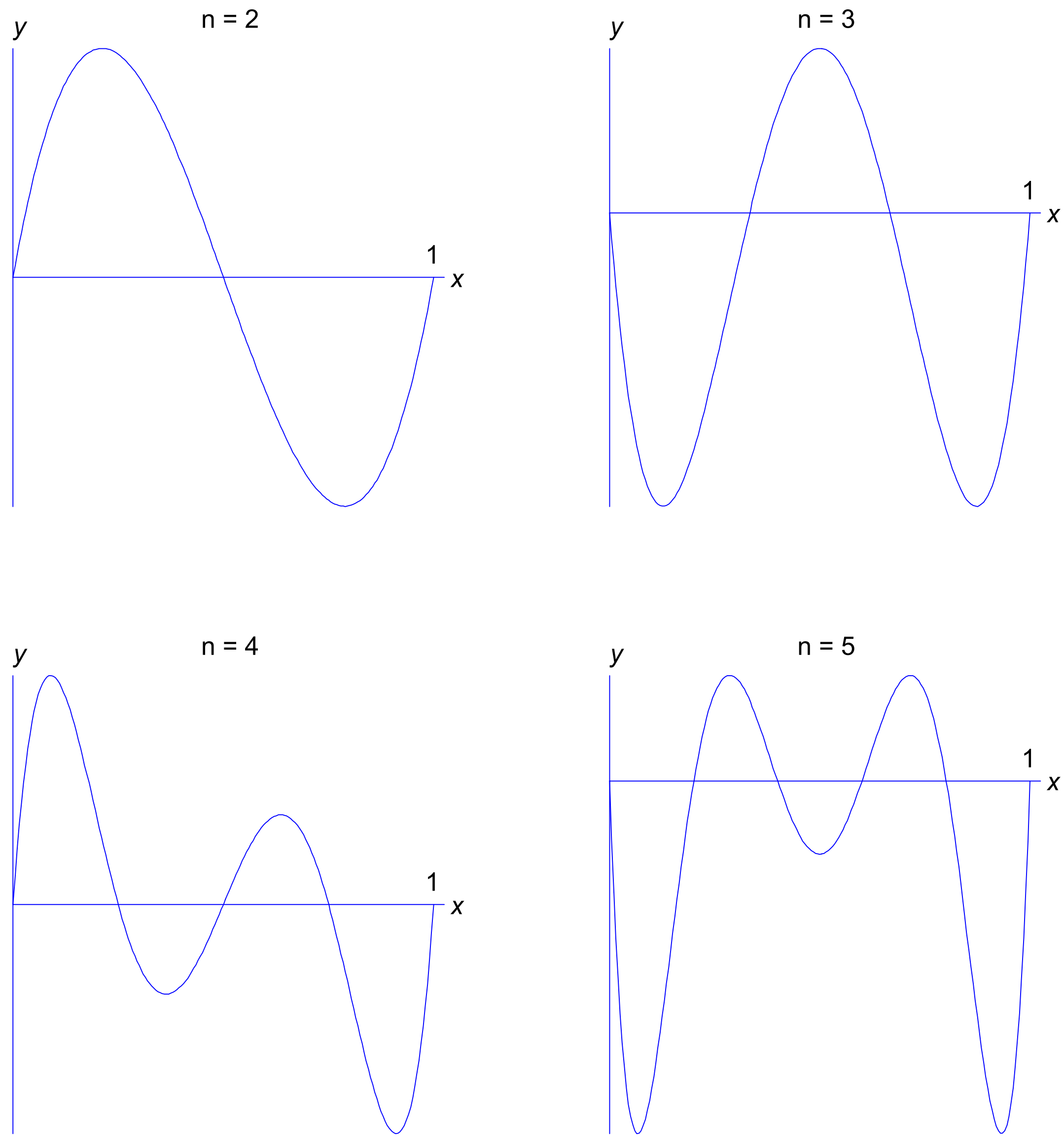
The interpolation nodes are determined by using

$$h = \frac{1}{n},\ x_0 = 0,\ x_1 = h,\ x_2 = 2h, ...,\ x_n = nh = 1$$

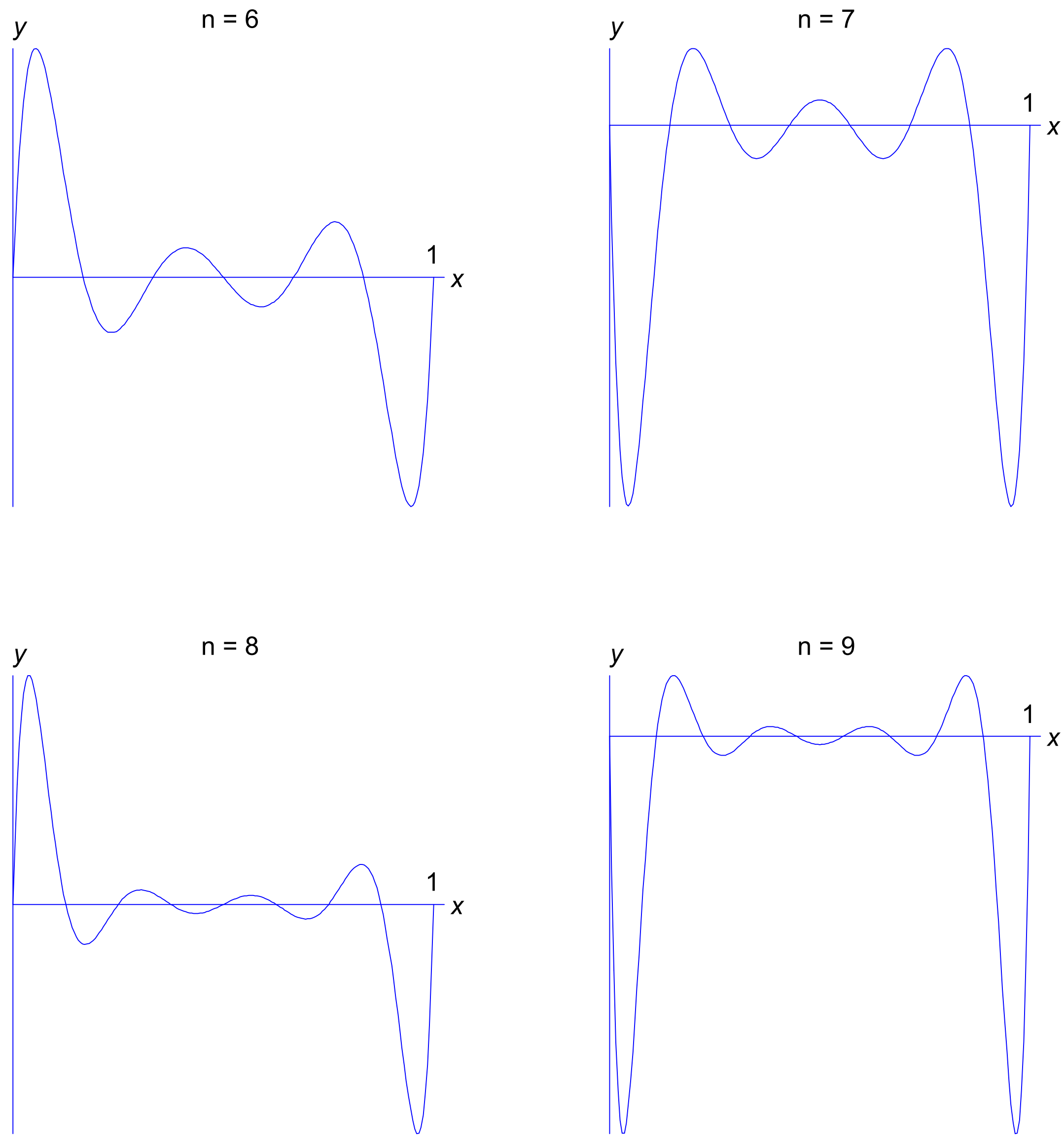
For this case,

$$\Psi_n(x) = x\,(x - h)\,(x - 2h) \cdots (x - 1)$$

Our graphs are the cases of $n = 2, ..., 9$.



Graphs of $\Psi_n(x)$ on $[0, 1]$ for $n = 2, 3, 4, 5$

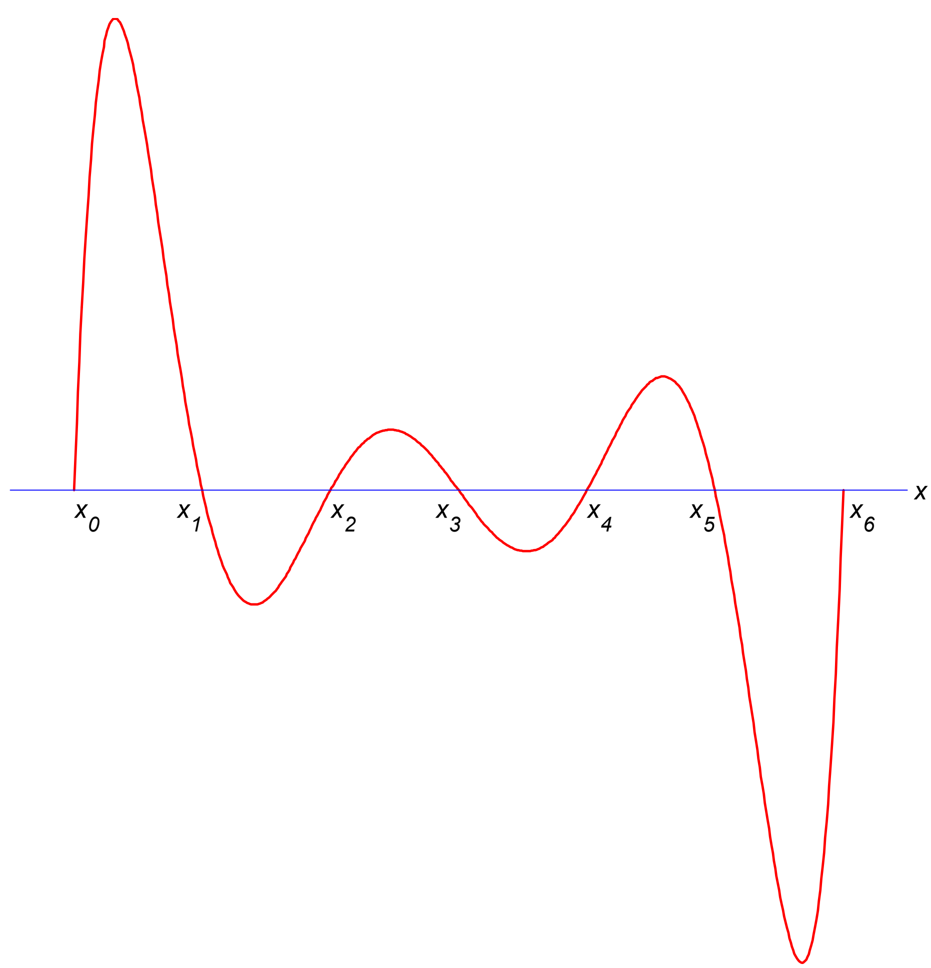


Graphs of $\Psi_n(x)$ on $[0,1]$ for $n = 6, 7, 8, 9$

Graph of

$$\Psi_6(x) = (x - x_0)\,(x - x_1) \cdots (x - x_6)$$

with evenly spaced nodes:



Using the following table

,

n	M_n	n	M_n
1	1.25E−1	6	4.76E−7
2	2.41E−2	7	2.20E−8
3	2.06E−3	8	9.11E−10
4	1.48E−4	9	3.39E−11
5	9.01E−6	10	1.15E−12

we can observe that the maximum

$$M_n \equiv \max_{x_0 \le x \le x_n} \frac{|\Psi_n(x)|}{(n+1)!}$$

becomes smaller with increasing n.

From the graphs, there is enormous variation in the size of $\Psi_n(x)$ as x varies over $[0, 1]$; and thus there is also enormous variation in the error as x so varies. For example, in the $n = 9$ case,

$$\max_{x_0 \le x \le x_1} \frac{|\Psi_n(x)|}{(n+1)!} = 3.39 \times 10^{-11}$$

$$\max_{x_4 \le x \le x_5} \frac{|\Psi_n(x)|}{(n+1)!} = 6.89 \times 10^{-13}$$

and the ratio of these two errors is approximately 49. Thus the interpolation error is likely to be around 49 times larger when $x_0 \le x \le x_1$ as compared to the case when $x_4 \le x \le x_5$. When doing table interpolation, the point x at which you are interpolating should be centrally located with respect to the interpolation nodes m$\{x_0, ..., x_n\}$ being used to define the interpolation, if possible.

AN APPROXIMATION PROBLEM

Consider now the problem of using an interpolation polynomial to approximate a given function $f(x)$ on a given interval $[a, b]$. In particular, take interpolation nodes

$$a \le x_0 < x_1 < \cdots < x_{n-1} < x_n \le b$$

and produce the interpolation polynomial $P_n(x)$ that interpolates $f(x)$ at the given node points. We would like to have

$$\max_{a \le x \le b} |f(x) - P_n(x)| \to 0 \quad \text{as} \quad n \to \infty$$

Does it happen?

Recall the error bound

$$\max_{a \le x \le b} |f(x) - P_n(x)| \\ \le \max_{a \le x \le b} \frac{|\Psi_n(x)|}{(n+1)!} \cdot \max_{a \le x \le b} \left| f^{(n+1)}(x) \right|$$

We begin with an example using evenly spaced node points.

RUNGE'S EXAMPLE

Use evenly spaced node points:

$$h = \frac{b-a}{n}, \qquad x_i = a + ih \quad \text{for} \quad i = 0, ..., n$$

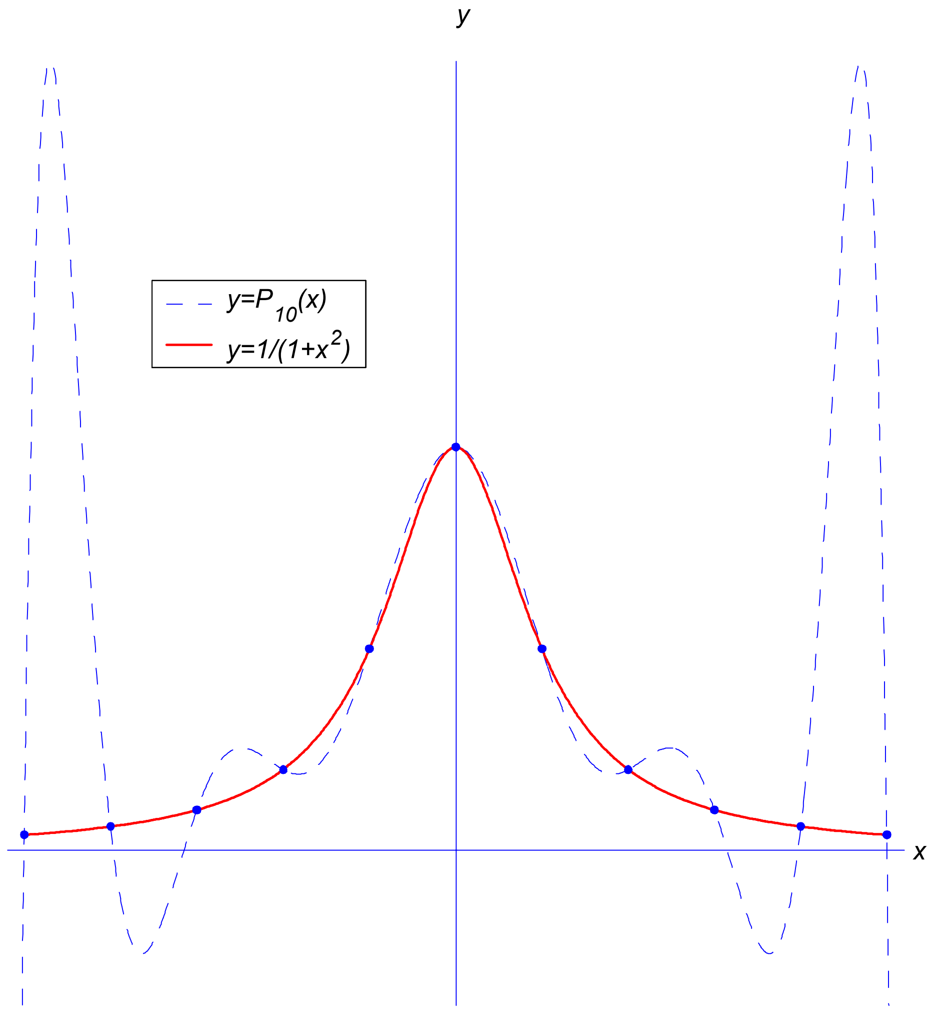
For some functions, such as $f(x) = e^x$, the maximum error goes to zero quite rapidly. But the size of the derivative term $f^{(n+1)}(x)$ in

$$\max_{a \le x \le b} |f(x) - P_n(x)|$$
$$\le \max_{a \le x \le b} \frac{|\Psi_n(x)|}{(n+1)!} \cdot \max_{a \le x \le b} \left| f^{(n+1)}(x) \right|$$

can badly hurt or destroy the convergence of other cases.

In particular, we show the graph of $f(x) = 1/\left(1 + x^2\right)$ and $P_n(x)$ on $[-5, 5]$ for the cases $n = 8$ and $n = 12$. The case $n = 10$ is in the text on page 127. It can be proven that for this function, the maximum error on $[-5, 5]$ does not converge to zero. Thus the use of evenly spaced nodes is not necessarily a good approach to approximating a function $f(x)$ by interpolation.

Runge's example with $n = 10$:



OTHER CHOICES OF NODES

Recall the general error bound

$$\max_{a \le x \le b} |f(x) - P_n(x)| \le \max_{a \le x \le b} \frac{|\Psi_n(x)|}{(n+1)!} \cdot \max_{a \le x \le b} \left| f^{(n+1)}(x) \right|$$

There is nothing we really do with the derivative term for f; but we can examine the way of defining the nodes $\{x_0, ..., x_n\}$ within the interval $[a, b]$. We ask how these nodes can be chosen so that the maximum of $|\Psi_n(x)|$ over $[a, b]$ is made as small as possible.

This problem has quite an elegant solution, and it is taken up in §4.6. The node points $\{x_0, ..., x_n\}$ turn out to be the zeros of a particular polynomial $T_{n+1}(x)$ of degree $n+1$, called a *Chebyshev polynomial*. These zeros are known explicitly, and with them

$$\max_{a \le x \le b} |\Psi_n(x)| = \left(\frac{b-a}{2}\right)^{n+1} 2^{-n}$$

This turns out to be smaller than for evenly spaced cases; and although this polynomial interpolation does not work for all functions $f(x)$, it works for all differentiable functions and more.

ANOTHER ERROR FORMULA

Recall the error formula

$$\begin{aligned} f(x) - P_n(x) &= \frac{\Psi_n(x)}{(n+1)!} f^{(n+1)}(c) \\ \Psi_n(x) &= (x - x_0)(x - x_1) \cdots (x - x_n) \end{aligned}$$

with c between the minimum and maximum of $\{x_0, ..., x_n, x\}$. A second formula is given by

$$f(x) - P_n(x) = \Psi_n(x)\, f[x_0, ..., x_n, x]$$

To show this is a simple, but somewhat subtle argument.

Let $P_{n+1}(x)$ denote the polynomial of degree $\leq n+1$ which interpolates $f(x)$ at the points $\{x_0, ..., x_n, x_{n+1}\}$. Then

$$\begin{aligned} P_{n+1}(x) &= P_n(x) \\ &\quad + f[x_0, ..., x_n, x_{n+1}]\,(x - x_0) \cdots (x - x_n) \end{aligned}$$

Substituting $x = x_{n+1}$, and using the fact that $P_{n+1}(x)$ interpolates $f(x)$ at x_{n+1}, we have

$$f(x_{n+1}) = P_n(x_{n+1}) \\ \qquad + f[x_0, ..., x_n, x_{n+1}]\,(x_{n+1} - x_0) \cdots (x_{n+1} - x_n)$$

$$f(x_{n+1}) = P_n(x_{n+1}) \\ \qquad + f[x_0, ..., x_n, x_{n+1}]\,(x_{n+1} - x_0) \cdots (x_{n+1} - x_n)$$

In this formula, the number x_{n+1} is completely arbitrary, other than being distinct from the points in $\{x_0, ..., x_n\}$. To emphasize this fact, replace x_{n+1} by x throughout the formula, obtaining

$$\begin{aligned} f(x) &= P_n(x) + f[x_0, ..., x_n, x]\,(x - x_0) \cdots (x - x_n) \\ &= P_n(x) + \Psi_n(x)\, f[x_0, ..., x_n, x] \end{aligned}$$

provided $x \neq x_0, ..., x_n$.

The formula

$$\begin{aligned} f(x) &= P_n(x) + f[x_0, ..., x_n, x]\,(x - x_0) \cdots (x - x_n) \\ &= P_n(x) + \Psi_n(x)\, f[x_0, ..., x_n, x] \end{aligned}$$

is easily true for x a node point. Provided $f(x)$ is differentiable, the formula is also true for x a node point.

This shows

$$f(x) - P_n(x) = \Psi_n(x)\, f[x_0, ..., x_n, x]$$

Compare the two error formulas

$$f(x) - P_n(x) = \Psi_n(x)\, f[x_0, ..., x_n, x]$$

$$f(x) - P_n(x) = \frac{\Psi_n(x)}{(n+1)!} f^{(n+1)}(c)$$

Then

$$\Psi_n(x)\, f[x_0, ..., x_n, x] = \frac{\Psi_n(x)}{(n+1)!} f^{(n+1)}(c)$$

$$f[x_0, ..., x_n, x] = \frac{f^{(n+1)}(c)}{(n+1)!}$$

for some c between the smallest and largest of the numbers in $\{x_0, ..., x_n, x\}$.

To make this somewhat symmetric in its arguments, let $m = n + 1$, $x = x_{n+1}$. Then

$$f[x_0, ..., x_{m-1}, x_m] = \frac{f^{(m)}(c)}{m!}$$

with c an unknown number between the smallest and largest of the numbers in $\{x_0, ..., x_m\}$. This was given in an earlier lecture where divided differences were introduced.