INTERPOLATION

Interpolation is a process of finding a formula (often a polynomial) whose graph will pass through a given set of points (x, y).

As an example, consider defining

$$x_0 = 0, \qquad x_1 = \frac{\pi}{4}, \qquad x_2 = \frac{\pi}{2}$$

and

$$y_i = \cos x_i, \qquad i = 0, 1, 2$$

This gives us the three points

$$(0, 1)\,, \quad \left(\frac{\pi}{4}, \frac{1}{\text{sqrt}(2)}\right), \quad \left(\frac{\pi}{2}, 0\right)$$

Now find a quadratic polynomial

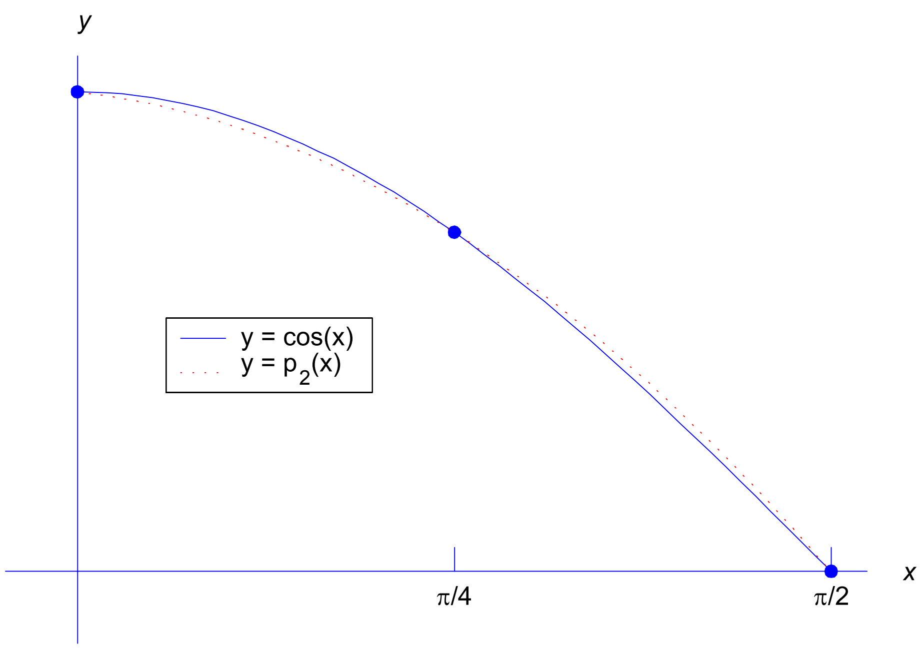
$$p(x) = a_0 + a_1 x + a_2 x^2$$

for which

$$p(x_i) = y_i, \quad i = 0, 1, 2$$

The graph of this polynomial is shown on the accompanying graph. We later give an explicit formula.

Quadratic interpolation of cos(x)



PURPOSES OF INTERPOLATION

1. Replace a set of data points $\{(x_i, y_i)\}$ with a function given analytically.

2. Approximate functions with simpler ones, usually polynomials or 'piecewise polynomials'.

Purpose #1 has several aspects.

- The data may be from a known class of functions. Interpolation is then used to find the member of this class of functions that agrees with the given data. For example, data may be generated from functions of the form

$$p(x) = a_0 + a_1 e^x + a_2 e^{2x} + \cdots + a_n e^{nx}$$

Then we need to find the coefficients $\left\{a_j\right\}$ based on the given data values.

- We may want to take function values $f(x)$ given in a table for selected values of x, often equally spaced, and extend the function to values of x not in the table.

  For example, given numbers from a table of logarithms, estimate the logarithm of a number x not in the table.

- Given a set of data points $\{(x_i, y_i)\}$, find a curve passing thru these points that is "pleasing to the eye". In fact, this is what is done continually with computer graphics. How do we connect a set of points to make a smooth curve? Connecting them with straight line segments will often give a curve with many corners, whereas what was intended was a smooth curve.

Purpose #2 for interpolation is to approximate functions $f(x)$ by simpler functions $p(x)$, perhaps to make it easier to integrate or differentiate $f(x)$. That will be the primary reason for studying interpolation in this course.

As as example of why this is important, consider the problem of evaluating

$$I = \int_0^1 \frac{dx}{1 + x^{10}}$$

This is very difficult to do analytically. But we will look at producing polynomial interpolants of the integrand; and polynomials are easily integrated exactly.

We begin by using polynomials as our means of doing interpolation. Later in the chapter, we consider more complex 'piecewise polynomial' functions, often called 'spline functions'.

LINEAR INTERPOLATION

The simplest form of interpolation is probably the straight line, connecting two points by a straight line. Let two data points (x_0, y_0) and (x_1, y_1) be given. There is a unique straight line passing through these points. We can write the formula for a straight line as

$$P_1(x) = a_0 + a_1 x$$

In fact, there are other more convenient ways to write it, and we give several of them below.

$$\begin{aligned}
P_1(x) &= \frac{x - x_1}{x_0 - x_1} y_0 + \frac{x - x_0}{x_1 - x_0} y_1 \\
&= \frac{(x_1 - x)\, y_0 + (x - x_0)\, y_1}{x_1 - x_0} \\
&= y_0 + \frac{x - x_0}{x_1 - x_0} \left[y_1 - y_0 \right] \\
&= y_0 + \left(\frac{y_1 - y_0}{x_1 - x_0} \right) (x - x_0)
\end{aligned}$$

Check each of these by evaluating them at $x = x_0$ and x_1 to see if the respective values are y_0 and y_1.

Example. Following is a table of values for $f(x) = \tan x$ for a few values of x.

x	1	1.1	1.2	1.3
$\tan x$	1.5574	1.9648	2.5722	3.6021

Use linear interpolation to estimate tan(1.15). Then use

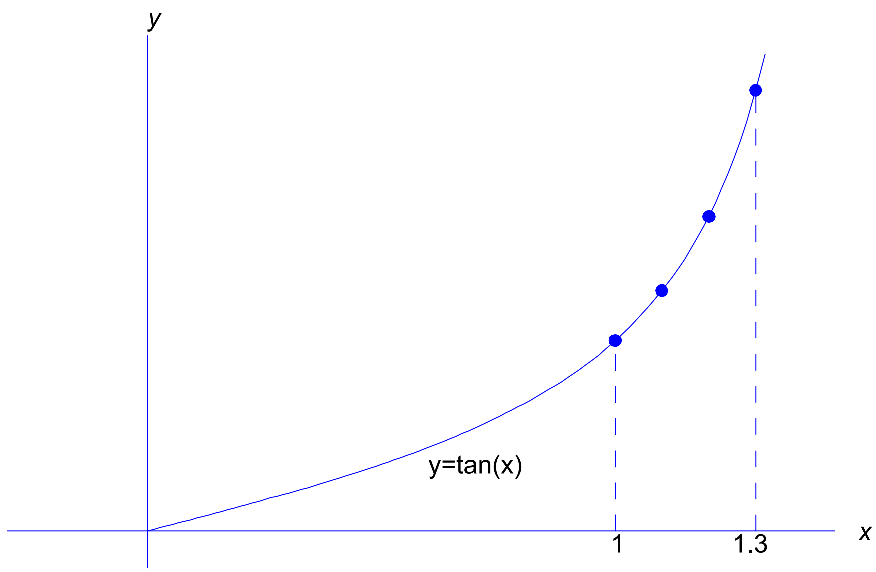
$$x_0 = 1.1, \quad x_1 = 1.2$$

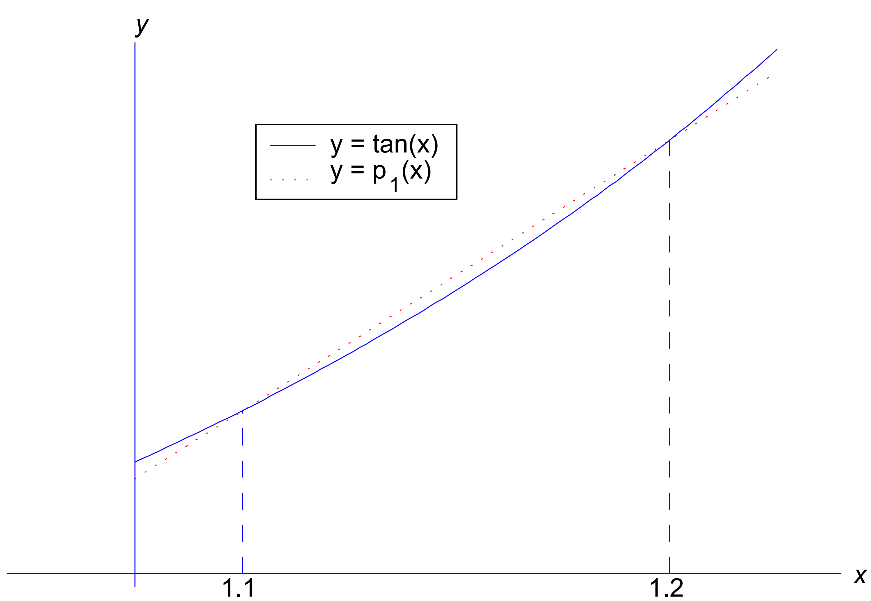
with corresponding values for y_0 and y_1. Then

$$\tan x \approx y_0 + \frac{x - x_0}{x_1 - x_0}\,[y_1 - y_0]$$

$$\begin{aligned}
\tan x &\approx y_0 + \frac{x - x_0}{x_1 - x_0}\,[y_1 - y_0] \\
\tan(1.15) &\approx 1.9648 + \frac{1.15 - 1.1}{1.2 - 1.1}\,[2.5722 - 1.9648] \\
&= 2.2685
\end{aligned}$$

The true value is $\tan 1.15 = 2.2345$. We will want to examine formulas for the error in interpolation, to know when we have sufficient accuracy in our interpolant.


y
y=tan(x)
1
1.3
x


y
y = tan(x)
y = p 1(x)
1.1
1.2
x

QUADRATIC INTERPOLATION

We want to find a polynomial

$$P_2(x) = a_0 + a_1 x + a_2 x^2$$

which satisfies

$$P_2(x_i) = y_i, \qquad i = 0, 1, 2$$

for given data points $(x_0, y_0), (x_1, y_1), (x_2, y_2)$. One formula for such a polynomial follows:

$$P_2(x) = y_0 L_0(x) + y_1 L_1(x) + y_2 L_2(x) \qquad (**)$$

with

$$L_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}, \qquad L_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}$$

$$L_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$

The formula $(**)$ is called *Lagrange's form* of the interpolation polynomial.

LAGRANGE BASIS FUNCTIONS

The functions

$$L_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}, \qquad L_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}$$

$$L_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$

are called 'Lagrange basis functions' for quadratic interpolation. They have the properties

$$L_i(x_j) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

for $i, j = 0, 1, 2$. Also, they all have degree 2. Their graphs are on an accompanying page.

As a consequence of each $L_i(x)$ being of degree 2, we have that the interpolant

$$P_2(x) = y_0 L_0(x) + y_1 L_1(x) + y_2 L_2(x)$$

must have degree ≤ 2.

UNIQUENESS

Can there be another polynomial, call it $Q(x)$, for which

$$\deg(Q) \le 2$$
$$Q(x_i) = y_i, \qquad i = 0, 1, 2$$

Thus, is the Lagrange formula $P_2(x)$ unique?

Introduce

$$R(x) = P_2(x) - Q(x)$$

From the properties of P_2 and Q, we have $\deg(R) \le 2$. Moreover,

$$R(x_i) = P_2(x_i) - Q(x_i) = y_i - y_i = 0$$

for all three node points x_0, x_1, and x_2. How many polynomials $R(x)$ are there of degree at most 2 and having three distinct zeros? The answer is that only the zero polynomial satisfies these properties, and therefore

$$R(x) = 0 \quad \text{for all } x$$

$$Q(x) = P_2(x) \quad \text{for all } x$$

SPECIAL CASES

Consider the data points

$$(x_0, 1), (x_1, 1), (x_2, 1)$$

What is the polynomial $P_2(x)$ in this case?

Answer: We must have the polynomial interpolant is

$$P_2(x) \equiv 1$$

meaning that $P_2(x)$ is the constant function. Why? First, the constant function satisfies the property of being of degree ≤ 2. Next, it clearly interpolates the given data. Therefore by the uniqueness of quadratic interpolation, $P_2(x)$ must be the constant function 1.

Consider now the data points

$$(x_0, mx_0), (x_1, mx_1), (x_2, mx_2)$$

for some constant m. What is $P_2(x)$ in this case? By an argument similar to that above,

$$P_2(x) = mx \quad \text{for all } x$$

Thus the degree of $P_2(x)$ can be less than 2.

HIGHER DEGREE INTERPOLATION

We consider now the case of interpolation by polynomials of a general degree n. We want to find a polynomial $P_n(x)$ for which

$$\begin{aligned} \deg(P_n) &\le n \\ P_n(x_i) = y_i, &\qquad i = 0, 1, \cdots, n \end{aligned} \qquad (**)$$

with given data points

$$(x_0, y_0), (x_1, y_1), \cdots, (x_n, y_n)$$

The solution is given by <u>Lagrange's formula</u>

$$P_n(x) = y_0 L_0(x) + y_1 L_1(x) + \cdots + y_n L_n(x)$$

The Lagrange basis functions are given by

$$L_k(x) = \frac{(x - x_0)\,..(x - x_{k-1})(x - x_{k+1})..\,(x - x_n)}{(x_k - x_0)\,..(x_k - x_{k-1})(x_k - x_{k+1})..\,(x_k - x_n)}$$

for $k = 0, 1, 2, ..., n$. The quadratic case was covered earlier.

In a manner analogous to the quadratic case, we can show that the above $P_n(x)$ is the only solution to the problem $(**)$.

In the formula

$$L_k(x) = \frac{(x - x_0)\,..(x - x_{k-1})(x - x_{k+1})..\,(x - x_n)}{(x_k - x_0)\,..(x_k - x_{k-1})(x_k - x_{k+1})..\,(x_k - x_n)}$$

we can see that each such function is a polynomial of degree n. In addition,

$$L_k(x_i) = \begin{cases} 1, & k = i \\ 0, & k \neq i \end{cases}$$

Using these properties, it follows that the formula

$$P_n(x) = y_0 L_0(x) + y_1 L_1(x) + \cdots + y_n L_n(x)$$

satisfies the interpolation problem of finding a solution to

$$\begin{gathered} \deg(P_n) \leq n \\ P_n(x_i) = y_i, \qquad i = 0, 1, \cdots, n \end{gathered}$$

EXAMPLE

Recall the table

x	1	1.1	1.2	1.3
$\tan x$	1.5574	1.9648	2.5722	3.6021

We now interpolate this table with the nodes

$$x_0 = 1,\ x_1 = 1.1,\ x_2 = 1.2,\ x_3 = 1.3$$

Without giving the details of the evaluation process, we have the following results for interpolation with degrees $n = 1, 2, 3$.

n	1	2	3
$P_n(1.15)$	2.2685	2.2435	2.2296
$Error$	$-.0340$	$-.0090$	.0049

It improves with increasing degree n, but not at a very rapid rate. In fact, the error becomes worse when n is increased further. Later we will see that interpolation of a much higher degree, say $n \geq 10$, is often poorly behaved when the node points $\{x_i\}$ are evenly spaced.

A FIRST ORDER DIVIDED DIFFERENCE

For a given function $f(x)$ and two distinct points x_0 and x_1, define

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

This is called a *first order divided difference* of $f(x)$.

By the Mean-value theorem,

$$f(x_1) - f(x_0) = f'(c)\,(x_1 - x_0)$$

for some c between x_0 and x_1. Thus

$$f[x_0, x_1] = f'(c)$$

and the divided difference in very much like the derivative, especially if x_0 and x_1 are quite close together. In fact,

$$f'\left(\frac{x_1 + x_0}{2}\right) \approx f[x_0, x_1]$$

is quite an accurate approximation of the derivative (see §5.4).

SECOND ORDER DIVIDED DIFFERENCES

Given three distinct points x_0, x_1, and x_2, define

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$

This is called the *second order divided difference* of $f(x)$.

By a fairly complicated argument, we can show

$$f[x_0, x_1, x_2] = \frac{1}{2} f''(c)$$

for some c intermediate to x_0, x_1, and x_2. In fact, as we investigate in §5.4,

$$f''(x_1) \approx 2f[x_0, x_1, x_2]$$

in the case the nodes are evenly spaced,

$$x_1 - x_0 = x_2 - x_1$$

EXAMPLE

Consider the table

x	1	1.1	1.2	1.3	1.4
$\cos x$	.54030	.45360	.36236	.26750	.16997

Let $x_0 = 1$, $x_1 = 1.1$, and $x_2 = 1.2$. Then

$$\begin{aligned} f[x_0, x_1] &= \frac{.45360 - .54030}{1.1 - 1} = -.86700 \\ f[x_1, x_2] &= \frac{.36236 - .45360}{1.1 - 1} = -.91240 \end{aligned}$$

$$\begin{aligned} f[x_0, x_1, x_2] &= \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} \\ &= \frac{-.91240 - (-.86700)}{1.2 - 1.0} = -.22700 \end{aligned}$$

For comparison,

$$\begin{aligned} f'\left(\frac{x_1 + x_0}{2}\right) &= -\sin(1.05) = -.86742 \\ \frac{1}{2} f''(x_1) &= -\frac{1}{2}\cos(1.1) = -.22680 \end{aligned}$$

GENERAL DIVIDED DIFFERENCES

Given $n+1$ distinct points $x_0, \ldots, x_n$, with $n \geq 2$, define

$$f[x_0, \ldots, x_n] = \frac{f[x_1, \ldots, x_n] - f[x_0, \ldots, x_{n-1}]}{x_n - x_0}$$

This is a recursive definition of the n^{th}-order divided difference of $f(x)$, using divided differences of order n. Its relation to the derivative is as follows:

$$f[x_0, \ldots, x_n] = \frac{1}{n!} f^{(n)}(c)$$

for some c intermediate to the points $\{x_0, \ldots, x_n\}$. Let I denote the interval

$$I = [\min\{x_0, \ldots, x_n\}, \max\{x_0, \ldots, x_n\}]$$

Then $c \in I$, and the above result is based on the assumption that $f(x)$ is n-times continuously differentiable on the interval I.

EXAMPLE

The following table gives divided differences for the data in

x	1	1.1	1.2	1.3	1.4
$\cos x$	.54030	.45360	.36236	.26750	.16997

For the column headings, we use

$$D^k f(x_i) = f[x_i, ..., x_{i+k}]$$

i	x_i	$f(x_i)$	$Df(x_i)$	$D^2 f(x_i)$	$D^3 f(x_i)$	$D^4 f(x_i)$
0	1.0	.54030	-.8670	-.2270	.1533	.0125
1	1.1	.45360	-.9124	-.1810	.1583	
2	1.2	.36236	-.9486	-.1335		
3	1.3	.26750	-.9753			
4	1.4	.16997				

These were computed using the recursive definition

$$f[x_0, ..., x_n] = \frac{f[x_1, ..., x_n] - f[x_0, ..., x_{n-1}]}{x_n - x_0}$$

ORDER OF THE NODES

Looking at $f[x_0, x_1]$, we have

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{f(x_0) - f(x_1)}{x_0 - x_1} = f[x_1, x_0]$$

The order of x_0 and x_1 does not matter. Looking at

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$

we can expand it to get

$$f[x_0, x_1, x_2] = \frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2)} + \frac{f(x_1)}{(x_1 - x_0)(x_1 - x_2)} + \frac{f(x_2)}{(x_2 - x_0)(x_2 - x_1)}$$

With this formula, we can show that the order of the arguments x_0, x_1, x_2 does not matter in the final value of $f[x_0, x_1, x_2]$ we obtain. Mathematically,

$$f[x_0, x_1, x_2] = f[x_{i_0}, x_{i_1}, x_{i_2}]$$

for any permutation (i_0, i_1, i_2) of $(0, 1, 2)$.

We can show in general that the value of $f[x_0, ..., x_n]$ is independent of the order of the arguments $\{x_0, ..., x_n\}$, even though the intermediate steps in its calculations using

$$f[x_0, ..., x_n] = \frac{f[x_1, ..., x_n] - f[x_0, ..., x_{n-1}]}{x_n - x_0}$$

are order dependent.

We can show

$$f[x_0, ..., x_n] = f[x_{i_0}, ..., x_{i_n}]$$

for any permutation $(i_0, i_1, ..., i_n)$ of $(0, 1, ..., n)$.

COINCIDENT NODES

What happens when some of the nodes $\{x_0, ..., x_n\}$ are not distinct. Begin by investigating what happens when they all come together as a single point x_0.

For first order divided differences, we have

$$\lim_{x_1 \to x_0} f[x_0, x_1] = \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0} = f'(x_0)$$

We extend the definition of $f[x_0, x_1]$ to coincident nodes using

$$f[x_0, x_0] = f'(x_0)$$

For second order divided differences, recall

$$f[x_0, x_1, x_2] = \frac{1}{2} f''(c)$$

with c intermediate to x_0, x_1, and x_2.

Then as $x_1 \rightarrow x_0$ and $x_2 \rightarrow x_0$, we must also have that $c \rightarrow x_0$. Therefore,

$$\lim_{\substack{x_1 \rightarrow x_0 \\ x_2 \rightarrow x_0}} f[x_0, x_1, x_2] = \frac{1}{2} f''(x_0)$$

We therefore define

$$f[x_0, x_0, x_0] = \frac{1}{2} f''(x_0)$$

For the case of general $f[x_0, ..., x_n]$, recall that

$$f[x_0, ..., x_n] = \frac{1}{n!} f^{(n)}(c)$$

for some c intermediate to $\{x_0, ..., x_n\}$. Then

$$\lim_{\{x1,...,x_n\} \to x_0} f[x_0, ..., x_n] = \frac{1}{n!} f^{(n)}(x_0)$$

and we define

$$f[\underbrace{x_0, ..., x_0}_{n+1 \text{ times}}] = \frac{1}{n!} f^{(n)}(x_0)$$

What do we do when only some of the nodes are coincident. This too can be dealt with, although we do so here only by examples.

$$\begin{aligned} f[x_0, x_1, x_1] &= \frac{f[x_1, x_1] - f[x_0, x_1]}{x_1 - x_0} \\ &= \frac{f'(x_1) - f[x_0, x_1]}{x_1 - x_0} \end{aligned}$$

The recursion formula can be used in general in this way to allow all possible combinations of possibly coincident nodes.

LAGRANGE'S FORMULA FOR THE INTERPOLATION POLYNOMIAL

Recall the general interpolation problem: find a polynomial $P_n(x)$ for which

$$\begin{aligned} \text{deg}(P_n) &\le n \\ P_n(x_i) &= y_i, \qquad i = 0, 1, \cdots, n \end{aligned}$$

with given data points

$$(x_0, y_0), (x_1, y_1), \cdots, (x_n, y_n)$$

and with $\{x_0, ..., x_n\}$ distinct points.

In §5.1, we gave the solution as *Lagrange's formula*

$$P_n(x) = y_0 L_0(x) + y_1 L_1(x) + \cdots + y_n L_n(x)$$

with $\{L_0(x), ..., L_n(x)\}$ the Lagrange basis polynomials. Each L_j is of degree n and it satisfies

$$L_j(x_i) = \begin{cases} 1, & j = i \\ 0, & j \ne i \end{cases}$$

for $i = 0, 1, ..., n$.

THE NEWTON DIVIDED DIFFERENCE FORM OF THE INTERPOLATION POLYNOMIAL

Let the data values for the problem

$$\begin{gathered}\deg(P_n) \le n \\ P_n(x_i) = y_i, \qquad i = 0, 1, \cdots, n\end{gathered}$$

be generated from a function $f(x)$:

$$y_i = f(x_i), \qquad i = 0, 1, ..., n$$

Using the divided differences

$$f[x_0, x_1],\ f[x_0, x_1, x_2], ..., f[x_0, ..., x_n]$$

we can write the interpolation polynomials

$$P_1(x),\ P_2(x), ..., P_n(x)$$

in a way that is simple to compute.

$$\begin{aligned} P_1(x) &= f(x_0) + f[x_0, x_1]\,(x - x_0) \\ P_2(x) &= f(x_0) + f[x_0, x_1]\,(x - x_0) \\ &\qquad + f[x_0, x_1, x_2]\,(x - x_0)\,(x - x_1) \\ &= P_1(x) + f[x_0, x_1, x_2]\,(x - x_0)\,(x - x_1) \end{aligned}$$

For the case of the general problem

$$\begin{gathered}\deg(P_n) \le n \\ P_n(x_i) = y_i, \qquad i = 0, 1, \cdots, n\end{gathered}$$

we have

$$\begin{aligned}P_n(x) \;=\;& f(x_0) + f[x_0, x_1]\,(x - x_0) \\ & + f[x_0, x_1, x_2]\,(x - x_0)\,(x - x_1) \\ & + f[x_0, x_1, x_2, x_3]\,(x - x_0)\,(x - x_1)\,(x - x_2) \\ & + \cdots \\ & + f[x_0, \ldots, x_n]\,(x - x_0)\cdots(x - x_{n-1})\end{aligned}$$

From this we have the recursion relation

$$P_n(x) = P_{n-1}(x) + f[x_0, \ldots, x_n]\,(x - x_0)\cdots(x - x_{n-1})$$

in which $P_{n-1}(x)$ interpolates $f(x)$ at the points in $\{x_0, \ldots, x_{n-1}\}$.

Example: Recall the table

i	x_i	$f(x_i)$	$Df(x_i)$	$D^2f(x_i)$	$D^3f(x_i)$	$D^4f(x_i)$
0	1.0	.54030	-.8670	-.2270	.1533	.0125
1	1.1	.45360	-.9124	-.1810	.1583	
2	1.2	.36236	-.9486	-.1335		
3	1.3	.26750	-.9753			
4	1.4	.16997				

with $D^k f(x_i) = f[x_i, ..., x_{i+k}], \quad k = 1, 2, 3, 4$. Then

$$\begin{aligned}
P_1(x) &= .5403 - .8670\,(x-1) \\
P_2(x) &= P_1(x) - .2270\,(x-1)\,(x-1.1) \\
P_3(x) &= P_2(x) + .1533\,(x-1)\,(x-1.1)\,(x-1.2) \\
P_4(x) &= P_3(x) \\
&\quad +.0125\,(x-1)\,(x-1.1)\,(x-1.2)\,(x-1.3)
\end{aligned}$$

Using this table and these formulas, we have the following table of interpolants for the value $x = 1.05$. The true value is cos(1.05) $=$ .49757105.

n	1	2	3	4
$P_n(1.05)$	.49695	.49752	.49758	.49757
$Error$	6.20E$-$4	5.00E$-$5	$-$1.00E$-$5	0.0

EVALUATION OF THE DIVIDED DIFFERENCE INTERPOLATION POLYNOMIAL

Let

$$\begin{aligned}
d_1 &= f[x_0, x_1] \\
d_2 &= f[x_0, x_1, x_2] \\
&\vdots \\
d_n &= f[x_0, ..., x_n]
\end{aligned}$$

Then the formula

$$\begin{aligned}
P_n(x) = \; & f(x_0) + f[x_0, x_1]\,(x - x_0) \\
& + f[x_0, x_1, x_2]\,(x - x_0)\,(x - x_1) \\
& + f[x_0, x_1, x_2, x_3]\,(x - x_0)\,(x - x_1)\,(x - x_2) \\
& + \cdots \\
& + f[x_0, ..., x_n]\,(x - x_0) \cdots (x - x_{n-1})
\end{aligned}$$

can be written as

$$\begin{aligned}
P_n(x) = \; & f(x_0) + (x - x_0)\,(d_1 + (x - x_1)\,(d_2 + \cdots \\
& + (x - x_{n-2})\,(d_{n-1} + (x - x_{n-1})\,d_n) \cdots)
\end{aligned}$$

Thus we have a nested polynomial evaluation, and this is quite efficient in computational cost.