THE TAYLOR POLYNOMIAL ERROR FORMULA

Let $f(x)$ be a given function, and assume it has derivatives around some point $x = a$ (with as many derivatives as we find necessary). For the error in the Taylor polynomial $p_n(x)$, we have the formulas

$$\begin{aligned} f(x) - p_n(x) &= \frac{1}{(n+1)!}(x-a)^{n+1} f^{(n+1)}(c_x) \\ &= \frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)\,dt \end{aligned}$$

The point c_x is restricted to the interval bounded by x and a, and otherwise c_x is unknown. We will use the first form of this error formula, although the second is more precise in that you do not need to deal with the unknown point c_x.

Consider the special case of $n = 0$. Then the Taylor polynomial is the constant function:

$$f(x) \approx p_0(x) = f(a)$$

The first form of the error formula becomes

$$f(x) - p_0(x) = f(x) - f(a) = (x - a)\, f'(c_x)$$

with c_x between a and x. You have seen this in your beginning calculus course, and it is called the mean-value theorem. The error formula

$$f(x) - p_n(x) = \frac{1}{(n+1)!}(x-a)^{n+1} f^{(n+1)}(c_x)$$

can be considered a generalization of the mean-value theorem.

EXAMPLE: $f(x) = e^x$

For general $n \geq 0$, and expanding e^x about $x = 0$, we have that the degree n Taylor polynomial approximation is given by

$$p_n(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n$$

For the derivatives of $f(x) = e^x$, we have

$$f^{(k)}(x) = e^x, \qquad f^{(k)}(0) = 1, \qquad k = 0, 1, 2, ...$$

For the error,

$$e^x - p_n(x) = \frac{1}{(n+1)!}x^{n+1}e^{c_x}$$

with c_x located between 0 and x. Note that for $x \approx 0$, we must have $c_x \approx 0$ and

$$e^x - p_n(x) \approx \frac{1}{(n+1)!}x^{n+1}$$

This last term is also the final term in $p_{n+1}(x)$, and thus

$$e^x - p_n(x) \approx p_{n+1}(x) - p_n(x)$$

Consider calculating an approximation to e. Then let $x = 1$ in the earlier formulas to get

$$p_n(1) = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$$

For the error,

$$e - p_n(1) = \frac{1}{(n+1)!} e^{c_x}, \qquad 0 \leq c_x \leq 1$$

To bound the error, we have

$$e^0 \leq e^{c_x} \leq e^1$$

$$\frac{1}{(n+1)!} \leq e - p_n(1) \leq \frac{e}{(n+1)!}$$

To have an approximation accurate to within 10^{-5}, we choose n large enough to have

$$\frac{e}{(n+1)!} \leq 10^{-5}$$

which is true if $n \geq 8$. In fact,

$$e - p_8(1) \leq \frac{e}{9!} \doteq 7.5 \times 10^{-6}$$

Then calculate $p_8(1) \doteq 2.71827877$, and $e - p_8(1) \doteq 3.06 \times 10^{-6}$.

FORMULAS OF STANDARD FUNCTIONS

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \frac{x^{n+1}}{1-x}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^m \frac{x^{2m}}{(2m)!} + (-1)^m \frac{x^{2m+2}}{(2m+2)!} \cos c_x$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^{m-1} \frac{x^{2m-1}}{(2m-1)!} + (-1)^m \frac{x^{2m+1}}{(2m+1)!} \cos c_x$$

with c_x between 0 and x.

OBTAINING TAYLOR FORMULAS

Most Taylor polynomials have been bound by other than using the formula

$$p_n(x) = f(a) + (x-a)f'(a) + \frac{1}{2!}(x-a)^2 f''(a) + \cdots + \frac{1}{n!}(x-a)^n f^{(n)}(a)$$

because of the difficulty of obtaining the derivatives $f^{(k)}(x)$ for larger values of k. Actually, this is now much easier, as we can use $Maple$ or $Mathematica$. Nonetheless, most formulas have been obtained by manipulating standard formulas; and examples of this are given in the text.

For example, use

$$e^t = 1 + t + \frac{1}{2!}t^2 + \frac{1}{3!}t^3 + \cdots + \frac{1}{n!}t^n + \frac{1}{(n+1)!}t^{n+1}e^{c_t}$$

in which c_t is between 0 and t. Let $t = -x^2$ to obtain

$$e^{-x^2} = 1 - x^2 + \frac{1}{2!}x^4 - \frac{1}{3!}x^6 + \cdots + \frac{(-1)^n}{n!}x^{2n} + \frac{(-1)^{n+1}}{(n+1)!}x^{2n+2}e^{-\xi_x}$$

Because c_t must be between 0 and $-x^2$, we have it must be negative. Thus we let $c_t = -\xi_x$ in the error term, with $0 \leq \xi_x \leq x^2$.