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AN EXISTENCE THEOREM FOR ABEL INTEGRAL EQUATIONS*

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Abstract. An existence and smoothness theorem is given for the Abel integral equation $\int_0^s K(s,t)f(t)(s^p - t^p)^{-\alpha}\,dt = g(s)$, $0 < s \leqq T$, with given $p > 0$ and $0 < \alpha < 1$. Particular attention is given to the behavior of $g(s)$ and $f(s)$ about $s = 0$.

1. Introduction. Consider the Abel integral equation

$$\int_0^s \frac{K(s,t)f(t)\,dt}{(s^p - t^p)^\alpha} = g(s), \qquad 0 < s \leqq T, \tag{1.1}$$

with given $p > 0$ and $0 < \alpha < 1$. To avoid degeneracy, we shall assume $K(s,s) \neq 0$ for $0 \leqq s \leqq T$. This is a classical equation, and it is obtained from a variety of mathematical and physical problems; see the bibliography of Noble [7].

In the past this equation has been examined case by case (for example, see Schmeidler [8] and the references in [7]). The methods of analysis were usually constructive or explicit, and the numerical analysis of (1.1) was usually based on these methods. Within the last few years, direct numerical methods for (1.1) have been proposed and studied (see [1]–[6], [10], [11]). These are general numerical methods which depend only on the smoothness of $K(s,t)$ and $f(t)$. As a complementary study to the numerical analysis of (1.1), we give a result on the existence and smoothness of solutions.

We shall need some special function spaces. For $\gamma > -1$, let us define

$$\mathscr{X}_\gamma = \{s^\gamma f(s) | f \in C[0,T]\},$$
$$\mathscr{X} = \bigcup_{\gamma > -1} \mathscr{X}_\gamma.$$

It can easily be seen that if $\gamma < \delta$, then $\mathscr{X}_\gamma \supset \mathscr{X}_\delta$. The space is much, but not all, of $L(0,T) \cap C(0,T]$.

THEOREM. *Let $g(s)$ have the form*

$$g(s) = s^\beta \tilde{g}(s), \quad 0 < s \leqq T, \quad \tilde{g} \in C^{n+1}[0,T], \tag{1.2}$$

for some integer $n \geqq 0$. Let β satisfy

$$p\alpha + \beta > 0. \tag{1.3}$$

Assume $K(s,t)$ is $n+2$ times continuously differentiable for $0 \leqq t \leqq s \leqq T$, and furthermore,

$$K(s,s) \neq 0, \qquad 0 \leqq s \leqq T. \tag{1.4}$$

Then there is a unique solution $f \in \mathscr{X}$ of (1.1), and its form is

$$f(s) = s^{p\alpha+\beta-1}[a + sl(s)] \equiv s^{p\alpha+\beta-1}\tilde{f}(s), \qquad s > 0, \tag{1.5}$$

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with $l \in C^n[0, T]$. *The constant* $a = 0$ *if and only if* $\tilde{g}(0) = 0$. (Note that the special form of $\tilde{f}(s)$ implies the existence of $\tilde{f}^{(n+1)}(0)$.) *Finally, there is a constant* $d_n > 0$, *independent of* $\tilde{g} \in C^n[0, T]$, *for which*

$$\max\{\|\tilde{f}\|, \cdots, \|\tilde{f}^{(n)}\|\} \leqq d_n \max\{\|\tilde{g}\|, \cdots, \|\tilde{g}^{(n+1)}\|\}. \tag{1.6}$$

The norm is the max norm on $[0, T]$.

In § 2 we give some standard results for $K(s, t) \equiv 1$. In § 3, we introduce a decomposition of (1.1) and prove some preliminary results about it. The proof of the theorem is given in § 4.

The theorem is true for systems as well. Let $K(s, t)$ be an $m \times m$ matrix, and let f and g be m-component column vectors. Condition (1.4) is replaced by

$$\det K(s, s) \neq 0, \qquad 0 \leqq s \leqq T;$$

all smoothness statements generalize immediately. The proof given in § 3 and § 4 generalizes by merely replacing absolute values by appropriate vector and matrix norms.

2. The Abel transform. Define the Abel transform by

$$\mathscr{A}h(s) = \int_0^s \frac{h(t)\,dt}{(s^p - t^p)^\alpha}, \qquad 0 < s \leqq T, \quad h \in L^1(0, T) \cap C(0, T].$$

(See Sneddon [9] for some properties and uses of the transform.) We give the needed properties of $\mathscr{A}$ in the following lemma.

LEMMA. *Consider the equation*

$$\int_0^s \frac{f(t)\,dt}{(s^p - t^p)^\alpha} = s^\beta \tilde{g}(s), \qquad 0 < s \leqq T, \tag{2.1}$$

with $\tilde{g} \in C^{n+1}[0, T]$, *for some* $n \geqq 0$ *and* $p\alpha + \beta > 0$. *Then there is a unique solution* $f \in L^1(0, T) \cap C(0, T]$ *and its form is*

$$f(s) = s^{p\alpha + \beta - 1}[a + sk(s)] \equiv s^{p\alpha+\beta-1}\tilde{f}(s), \tag{2.2}$$

with $k \in C^n[0, T]$ *and* $a \equiv$ const. *Moreover, for some constant* d,

$$\|\tilde{f}^{(n)}\| \leqq d \max\{\|\tilde{g}\|, \cdots, \|\tilde{g}^{(n+1)}\|\}. \tag{2.3}$$

Proof. The inverse of $\mathscr{A}$ is given by

$$\mathscr{A}^{-1}h(s) = \frac{p \sin(\alpha\pi)}{\pi} \frac{d}{ds} \int_0^s \frac{r^{p-1}h(r)\,dr}{(s^p - r^p)^{1-\alpha}}, \qquad s > 0. \tag{2.4}$$

Using this and a change of the variable of integration, we obtain (2.2) with

$$\begin{aligned} a &= \frac{p \sin(\alpha\pi)}{\pi}(p\alpha + \beta)\tilde{g}(0) \int_0^1 \frac{u^{p+\beta-1}\,du}{(1 - u^p)^{1-\alpha}}, \\ k(s) &= \frac{p \sin(\alpha\pi)}{\pi} \int_0^1 \frac{u^{p+\beta}}{(1 - u^p)^{1-\alpha}} \left[\tilde{g}'(us) + (p\alpha + \beta)\frac{\tilde{g}(us) - \tilde{g}(0)}{us}\right] du. \end{aligned} \tag{2.5}$$

The remaining results follow from the formula for $k(s)$.

3. Preliminary definitions and results. Define

$$\mathscr{K}h(s) = \int_0^s \frac{K(s,t)f(t)\,dt}{(s^p - t^p)^\alpha}, \qquad 0 < s \leqq T, \quad h \in L^1(0,T) \cap C(0,T].$$

To simplify some formulas, we assume, without loss of generality, that

$$K(s,s) = 1, \qquad 0 \leqq s \leqq T.$$

Assuming $K_2(s,s) = \partial K(s,t)/\partial t|_{t=s}$ exists, let us define

$$H(s,t) = \begin{cases} \dfrac{K(s,s) - K(s,t)}{s - t}, & s > t, \\ \dfrac{\partial K(s,t)}{\partial t}, & s = t; \end{cases}$$

$$\mathscr{H}h(s) = \int_0^s \frac{H(s,t)(s-t)^{1-\alpha}h(t)\,dt}{[(s^p - t^p)/(s -)]^\alpha}, \qquad 0 < s \leqq T.$$

Then

$$\mathscr{K} = \mathscr{A} - \mathscr{H}. \tag{3.1}$$

To solve $\mathscr{K}f = g$, equation (1.1), we solve the problem

$$\mathscr{A}z = g, \quad f - \mathscr{A}^{-1}\mathscr{H}f = z. \tag{3.2}$$

To examine the existence and smoothness of f, we shall need a formula for $\mathscr{A}^{-1}\mathscr{H}$. An especially useful one is

$$\begin{aligned} \mathscr{A}^{-1}\mathscr{H}h(s) = \frac{sp\sin(\alpha\pi)}{\pi}\Bigg\{&\int_0^1 \frac{u^{p-1}}{(1-u^p)^{1-\alpha}}\int_0^u \frac{h(ws)(u-w)^{1-\alpha}}{[(u^p - w^p)/(u-w)]^\alpha}[p\alpha H(us,ws) \\ &\qquad + usH_1(us,ws)]\,dw\,du \\ &+ \int_0^1 \frac{u^p}{(1-u^p)^{1-\alpha}}\int_0^u \frac{H(us,ws)h(ws)}{(u^p - w^p)^\alpha} \\ &\qquad \cdot \left[1 - \frac{\alpha p w^{p-1}(u-w)}{u^p - w^p}\right] dw\,du\Bigg\}, \end{aligned} \tag{3.3}$$

which is valid for all $h \in \mathscr{X}$. To obtain it, we take a specific form for h, say $h(s) = s^\gamma \tilde{h}(s)$, for some $\gamma > -1$, $\tilde{h} \in C[0,T]$. Substituting this into $\mathscr{H}h(s)$, we make a change of variable, and note the behavior of $\mathscr{H}h(s)$ about $s = 0$. We substitute this into (2.4), and then perform much algebraic manipulation to obtain (3.3). Note that we need the existence of the partial derivative $H_1(s,t)$, which follows from the fact that $K(s,t)$ is twice continuously differentiable.

We also need a number of special inequalities. From the identity

$$\frac{1-s^p}{1-s} = p\int_0^1 [1 - (1-s)r]^{p-1}\,dr, \qquad 0 \leqq s < 1, \quad p > 0,$$

we obtain

$$\min\{1,p\} \leqq \frac{1-s^p}{1-s} \leqq \max\{1,p\}, \qquad 0 \leqq s < 1, \quad p > 0. \tag{3.4}$$

From the estimate

$$\Gamma(x+1) = \sqrt{2\pi} x^{x+1/2} e^{-x+\theta/12x}, \qquad x > 0, \quad 0 < \theta < 1,$$

we obtain

$$\frac{\Gamma(x)}{\Gamma(x+\lambda)} \leqq \frac{\gamma(x)}{(x+\lambda)^\lambda}, \qquad x > 0, \quad 0 \leqq \lambda \leqq 1, \tag{3.5}$$

with

$$\gamma(x) = \left(1 + \frac{1}{x}\right) e^{1+1/12x},$$

a monotone decreasing function of x on $(0, \infty)$.

Define

$$\begin{aligned} A(l) &= \int_0^1 \frac{u^{p-1}}{(1-u^p)^{1-\alpha}} \int_0^u w^l (u-w)^{1-\alpha} \left[\frac{u-w}{u^p - w^p}\right]^\alpha dw\, du, \\ B(l) &= \int_0^1 \frac{u^p}{(1-u^p)^{1-\alpha}} \int_0^u \frac{w^l\, dw}{(u^p - w^p)^\alpha} du, \qquad l > -1. \end{aligned} \tag{3.6}$$

We use the change of variable $w = uv$, $0 \leqq v \leqq 1$, the bounds (3.4), and some manipulation to reduce (3.6) to new formulas involving beta functions. We evaluate these and then bound them, using (3.5), to obtain eventually

$$A(l) \leqq \frac{C_A(l)}{(l+2-\alpha)^2}, \quad B(l) \leqq \frac{C_B(l)}{(l+2-\alpha)}, \qquad l > -1, \tag{3.7}$$

with $C_A(l)$ and $C_B(l)$ monotone decreasing functions on $(-1, \infty)$.

4. Proof of theorem. The proof is divided into several parts.

(i) *Existence and uniqueness of solution $f \in \mathscr{X}$.* Recall the statement of the theorem. It is easily seen that if either $\mathscr{K} f = g$ or formulation (3.2) has a unique solution for a g of form (1.2), then so does the other. We shall use (3.2).

Let $\mathscr{A} z = g$. By the lemma,

$$z(s) = s^{p\alpha+\beta-1}[a + sk(s)] \equiv s^{p\alpha+\beta-1}\tilde{z}(s), \qquad k, \tilde{z} \in C^n[0, T]. \tag{4.1}$$

To show the unique solvability in $\mathscr{X}$ of $(I - \mathscr{A}^{-1}\mathscr{K})f = z$, we shall show that

$$I - \mathscr{A}^{-1}\mathscr{K} : \mathscr{X} \xrightarrow[\text{onto}]{1\text{-}1} \mathscr{X}.$$

This will be shown by proving that

$$I - \mathscr{A}^{-1}\mathscr{K} : \mathscr{X}_\gamma \xrightarrow[\text{onto}]{1\text{-}1} \mathscr{X}_\gamma \quad \text{for all } \gamma > -1. \tag{4.2}$$

From (4.2) and (4.1), we shall also have

$$f(s) = s^{p\alpha+\beta-1}\tilde{f}(s), \qquad \tilde{f} \in C[0, T]. \tag{4.3}$$

To prove that (4.2) holds, we begin by looking at (3.3) with $h(s) = s^\gamma \tilde{h}(s)$. Then

$$\begin{aligned}\mathscr{A}^{-1}\mathscr{H}h(s) = \frac{ps^{\gamma+1}\sin(\alpha\pi)}{\pi}\Bigg\{\int_0^1 \frac{u^{p-1}}{(1-u^p)^{1-\alpha}} \int_0^u \frac{w^\gamma \tilde{h}(ws)(u-w)^{1-\alpha}}{[(u^p-w^p)/(u-w)]^\alpha} \\ \cdot [p\alpha H(us, ws) + usH_1(us, ws)]\, dw\, du \\ + \int_0^1 \frac{u^p}{(1-u^p)^{1-\alpha}} \int_0^u \frac{H(us, ws)w^\gamma \tilde{h}(ws)}{(u^p-w^p)^\alpha}\left[1 - \frac{\alpha p w^{p-1}(u-w)}{u^p-w^p}\right] dw\, du\Bigg\} \\ \equiv s^{\gamma+1}\tilde{y}(s), \qquad \tilde{y} \in C[0, T].\end{aligned} \tag{4.4}$$

Thus

$$\mathscr{A}^{-1}\mathscr{H} : \mathscr{X}_\gamma \to \mathscr{X}_{\gamma+1}, \qquad \gamma > -1, \tag{4.5}$$

and this proves that $I - \mathscr{A}^{-1}\mathscr{H}$ maps $\mathscr{X}_\gamma$ into $\mathscr{X}_\gamma$.

Let $z \in \mathscr{X}_\gamma$ for some $\gamma > -1$, $z(s) = s^\gamma \tilde{z}(s)$. We shall show the existence of $f \in \mathscr{X}_\gamma$ with $(I - \mathscr{A}^{-1}\mathscr{H})f = z$ by looking at the Neumann series for the equation. Define

$$f_j = [\mathscr{A}^{-1}\mathscr{H}]^j z, \quad \tilde{f}_j(s) = s^{-\gamma} f_j(s), \qquad s \geqq 0, \quad j = 0, 1, 2, \cdots.$$

By induction, using (4.5), we have $f_j \in \mathscr{X}_\gamma$ for all $j \geqq 0$, and thus $\tilde{f}_j \in C[0, T]$. We shall show that

$$\tilde{f}(s) \equiv \sum_0^\infty \tilde{f}_j(s) \tag{4.6}$$

converges uniformly on $[0, T]$. It will follow by standard arguments for Neumann series that $f(s) \equiv s^\gamma \tilde{f}(s)$ is a solution of $(I - \mathscr{A}^{-1}\mathscr{H})f = z$. We shall discuss uniqueness later.

Let M be a bound on $|H(s, t)|$ and $|H_1(s, t)|$ for $0 \leqq t \leqq s \leqq T$. As an induction hypothesis, assume that for $i \leqq j$,

$$|\tilde{f}_i(s)| \leqq D_i s^i, \qquad 0 \leqq s \leqq T. \tag{4.7}$$

This is easily seen to be true for $j = 0$ since $\tilde{f}_0 \equiv \tilde{z}$; use $D_0 = \|\tilde{z}\|$. Assuming the hypothesis for a general j, we shall use (4.4) to prove it for $j + 1$. Since $\tilde{f}_{j+1}(s) = s^{-\gamma}[\mathscr{A}^{-1}\mathscr{H}f_j](s)$, from (4.4), (4.7), (3.4), and (3.6) it follows that

$$\begin{aligned}|\tilde{f}_{j+1}(s)| \leqq s^{j+1}\frac{p\sin(\alpha\pi)}{\pi} MD_j \Bigg\{(p\alpha + T)A(j+\gamma) \\ + \left(1 + p\alpha \max\left\{1, \frac{1}{p}\right\}\right)B(j+\gamma)\Bigg\}.\end{aligned}$$

From (3.7), with $r = \gamma + 2 - \alpha > 0$, we obtain

$$(4.8) \qquad |\tilde{f}_{j+1}(s)| \leqq D_{j+1} s^{j+1}, \quad D_{j+1} = \frac{C_0 D_j}{j + r},$$

$$C_0 = \frac{p \sin(\alpha\pi)}{\pi} M \left\{ (p\alpha + T) \frac{C_A(\gamma)}{1 - \alpha} + (1 + p\alpha \max\{1, 1/p\}) C_B(\gamma) \right\}.$$

The constant C_0 is independent of $j \geqq 0$. Also, the induction is completed.

Using (4.8) and $D_0 = \|\tilde{z}\|$, we obtain

$$(4.9) \qquad |\tilde{f}_j(s)| \leqq \Gamma(r) \|\tilde{z}\| \frac{C_0^j s^j}{\Gamma(r + j)}, \qquad j \geqq 0, \quad 0 \leqq s \leqq T.$$

For the series (4.6),

$$(4.10) \qquad |\tilde{f}(s)| \leqq \Gamma(r) \|\tilde{z}\| \sum_0^\infty \frac{C_0^j s^j}{\Gamma(r + j)}.$$

This converges uniformly on $[0, T]$, and thus $\tilde{f}(s)$ is continuous.

To prove the uniqueness in $\mathscr{X}_\gamma$ of the previously constructed f, let us assume that

$$y - \mathscr{A}^{-1}\mathscr{H} y = 0, \quad y(s) = s^\gamma \tilde{y}(s), \qquad \tilde{y} \in C[0, T].$$

Then

$$y = [\mathscr{A}^{-1}\mathscr{H}]^j y, \qquad j \geqq 0.$$

Applying the same kind of derivation as that used to obtain (4.8), with $\tilde{z}$ replaced by $\tilde{y}$, we obtain

$$|\tilde{y}(s)| \leqq \Gamma(r) \|\tilde{y}\| \frac{C_0^j s^j}{\Gamma(r + j)}, \qquad 0 \leqq s \leqq T, \quad j \geqq 0.$$

It follows that $\tilde{y} \equiv 0$, and thus $y \equiv 0$.

We combine (4.9) with (2.3) of the lemma to obtain the stability result (1.6) for the case $n = 0$. The proof of the remaining part of (1.5) is given later.

(ii) *Case* $n = 1$. We shall show that each $\tilde{f}_j \in C^1[0, T]$ and that

$$(4.11) \qquad \sum_0^\infty \tilde{f}_j'(s)$$

converges uniformly on $[0, T]$. It then follows by standard arguments that $\tilde{f} \in C^1[0, T]$ and that $\tilde{f}'(s)$ equals the series (4.11).

From $\tilde{f}_0 = \tilde{z}_0$ and (4.1), we have that $\tilde{f}_0 \in C^1[0, T]$. By induction on j using (4.4), it follows that $\tilde{f}_j \in C^1[0, T]$ for all j. For a second induction, assume that for $1 \leqq i \leqq j$,

$$|\tilde{f}_i'(s)| \leqq D_i^{(1)} s^{i-1}, \qquad 0 \leqq s \leqq T.$$

This is true for $j = 1$ since $\tilde{f}_1'(s)$ is continuous. Let us assume it for general j, and use (4.8) and the derivative $\tilde{f}'_{j+1}(s)$ from (4.4) to obtain

$$|\tilde{f}'_{j+1}(s)| \leqq C_1\left[\frac{C_1^j\Gamma(r)}{\Gamma(r+j)} + \frac{D_j^{(1)}}{r+j+1}\right]s^j = D_{j+1}^{(1)}s^j, \tag{4.12}$$

with $C_1 = \text{const.}$, $C_1 \geqq C_0$. The induction is completed. Also choose C_1 large enough to ensure that

$$|\tilde{f}_1(s)| \leqq C_1 \max\{\|\tilde{z}\|, \|\tilde{z}'\|\}.$$

From (4.12), it follows that

$$|\tilde{f}'_j(s)| \leqq \gamma_1(j)\frac{C_1^{j-1}s^{j-1}}{\Gamma(r+j-1)}\max\{\|\tilde{z}\|, \|\tilde{z}'\|\},$$

with $\gamma_1(j)$ a linear polynomial in j, for $j \geqq 1$, $0 \leqq s \leqq T$. From this it follows that the series of (4.11) converges uniformly, concluding the proof. The stability result (1.6) follows as with $n = 0$.

(iii) *A brief sketch of the general case.* Let us assume that the result has been proven for $n \leqq m-1$ and let us prove it for $n = m$. As part of the induction, we assume that

$$|\tilde{f}_j^{(n)}(s)| \leqq \gamma_n(j)\frac{C_1^{j-n}s^{j-n}}{\Gamma(r+j-n)}\max\{\|\tilde{z}\|, \cdots, \|\tilde{z}^{(n)}\|\} \tag{4.13}$$

for $j \geqq n$, $0 \leqq n \leqq m-1$, $0 \leqq s \leqq T$, with $\gamma_n(j)$ a polynomial in j of degree $\leqq n$. To prove the theorem for $n = m$, let us form the mth derivative of $\tilde{f}_{j+1}(s)$ using (4.4) and Leibniz's rule. Then we proceed exactly as with the case $n = 1$. The many details are omitted.

(iv) *The special form of* (1.5). Since $\tilde{f} \in C^n[0, T]$, we use $f(s) = s^{p\alpha+\beta-1}\tilde{f}(s)$ and (4.4) to obtain

$$\mathscr{A}^{-1}\mathscr{H}f(s) = s^{p\alpha+\beta}l(s), \qquad l \in C^n[0, T].$$

Using $f = z + \mathscr{A}^{-1}\mathscr{H}f$, formula (4.1), and the preceding equality we obtain

$$f(s) = s^{p\alpha+\beta-1}[a + s(k(s) + l(s))],$$

the desired form. From (2.5), it is seen that the constant $a = 0$ if and only if $\tilde{g}(0) = 0$.

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