

# Type Preservation as a Confluence Problem

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# The Standard Approach to Typing

Syntax.

$$\begin{aligned} \text{types } T &::= A \mid T_1 \Rightarrow T_2 \\ \text{terms } t &::= x \mid t_1 t_2 \mid \lambda x : T. t \end{aligned}$$

Operational semantics.

$t \rightarrow_c t'$ , small-step reduction relation (concrete).

Typing relation.

$\Gamma \vdash t : T$ , with  $\Gamma$  a *context*.

Type Preservation.

$$\Gamma \vdash t : T \wedge t \rightarrow_c t' \implies \Gamma \vdash t' : T$$

## Alternative: The Rewriting Approach to Typing

Proposed by Kuan, MacQueen, and Findler.

“A Rewriting Semantics for Type Inference”, ESOP 2007.

Define typing as small-step abstract reduction  $\rightarrow_a$ .

Terms rewrite to their types.

$$\begin{array}{ll} \lambda y : A.y & \rightarrow_a^* A \Rightarrow A \\ (\lambda x : A \Rightarrow A.x) \lambda y : A.y & \rightarrow_a^* A \Rightarrow A \\ \lambda x : A.\lambda y : A.x & \rightarrow_a^* A \Rightarrow A \Rightarrow A \end{array}$$

Formulate metatheory in term-rewriting terms.

## This Work

Build on Kuan et al.'s work by applying term-rewriting methods.

Prove confluence of combined reduction  $\rightarrow_{ca} = \rightarrow_c \cup \rightarrow_a$ .

Use decreasing diagrams.

Type Preservation is a corollary.

**Ultimate goal** : simplify metatheory of programming languages.

# The Rewriting Approach to STLC

# Syntax

To rewrite a term to type, need intermediate **mixed terms**.

$$\begin{aligned} \text{types } T & ::= A \mid T_1 \Rightarrow T_2 \\ \text{mixed terms } m & ::= x \mid m m' \mid \lambda x : T. m \mid A \mid T \Rightarrow m \\ \text{mixed values } u & ::= \lambda x : T. m \mid T \Rightarrow m \mid A \end{aligned}$$

$T \Rightarrow m$  is viewed as an **abstract function**.

Can be applied to arguments (concrete or abstract).

Example mixed terms:

$$\begin{aligned} & \lambda x : A. x \\ & A \Rightarrow A \\ & A \Rightarrow \lambda x : A. x \\ & (\lambda x : A. x) A \\ & (A \Rightarrow A) A \end{aligned}$$

## Concrete and Abstract Reduction Relations

Concrete reduction ( $\rightarrow_c$ ) is call-by-value.

Abstract reduction ( $\rightarrow_a$ ) is nondeterministic.

$$\frac{}{E_c[(\lambda x : T. m) u] \rightarrow_c E_c[[u/x]m]} c(\beta)$$

$$\frac{}{E_a[\lambda x : T. m] \rightarrow_a E_a[T \Rightarrow [T/x]m]} a(\lambda)$$

$$\frac{}{E_a[(T \Rightarrow m) T] \rightarrow_a E_a[m]} a(\beta)$$

where:

$$E_c ::= * \mid (E_c t) \mid (u E_c)$$

$$E_a ::= * \mid (E_a m) \mid (m E_a) \mid \lambda x : T. E_a \mid T \Rightarrow E_a$$

## An Example Abstract Reduction Sequence

Reduce a term to its type (redexes underlined):

$$\begin{aligned} & \lambda x : (A \Rightarrow A). \lambda y : A. \underline{(x (x y))} \rightarrow_a \\ & \lambda x : (A \Rightarrow A). \underline{A \Rightarrow (x (x A))} \rightarrow_a \\ & \underline{(A \Rightarrow A) \Rightarrow A \Rightarrow ((A \Rightarrow A) ((A \Rightarrow A) A))} \rightarrow_a \\ & (A \Rightarrow A) \Rightarrow A \Rightarrow \underline{((A \Rightarrow A) A)} \rightarrow_a \\ & (A \Rightarrow A) \Rightarrow A \Rightarrow A \end{aligned}$$

Use  $a(\lambda)$  rule to turn  $\lambda$  to  $\Rightarrow$ .

$$\lambda y : A. (x (x y)) \rightarrow_a A \Rightarrow (x (x A))$$

Use  $a(\beta)$  rule to reduce applications of abstract functions.

$$((A \Rightarrow A) A) \rightarrow_a A$$



## Metatheory: Relation to Standard Typing

Define standard terms  $t$ .

$$t ::= x \mid \lambda x : T. t \mid t t'$$

**Theorem.**  $x_1 : T_1, \dots, x_n : T_n \vdash t : T$  iff  $[T_1/x_1, \dots, T_n/x_n]t \rightarrow_a^* T$ .

Proof is similar to relating big-step and small-step concrete reduction.

Usual typing rules define big-step abstract reduction:

$$\frac{\Gamma \vdash t : T' \Rightarrow T \quad \Gamma \vdash t' : T'}{\Gamma \vdash t t' : T}$$

# Metatheory: Type Preservation

Standard formulation:

$$\Gamma \vdash t : T \wedge t \rightarrow_c t' \implies \Gamma \vdash t' : T$$

Rewriting formulation:

$$m \rightarrow_a^* T \wedge m \rightarrow_c m' \implies m' \rightarrow_a^* T$$

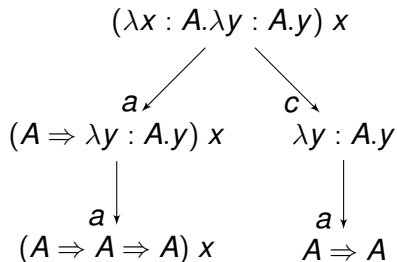
Follows from confluence of  $\rightarrow_{ca}$  for typable terms (prove now).

# Confluence of Combined Reduction on Typable Terms

## Initial Observations

Call  $m$  **typable** iff  $m \rightarrow_a^* T$ , for some  $T$ .

Confluence fails for non-typable terms.



We will prove confluence for typable terms.

# Basic Properties of Abstract Reduction

**Theorem.**  $\rightarrow_a$  is terminating.

**Theorem.**  $\rightarrow_a$  has the diamond property.

Proof:

No critical overlap of  $\rightarrow_a$ -redexes.

No duplication or deletion of  $\rightarrow_a$ -redexes.

$$\overline{E_a[\lambda x : T. m] \rightarrow_a E_a[T \Rightarrow [T/x]m]} \quad a(\lambda)$$

$$\overline{E_a[(T \Rightarrow m) T] \rightarrow_a E_a[m]} \quad a(\beta)$$

## Recall: Decreasing Diagrams [van Oostrom]

Assign labels to steps  $e \rightarrow e'$ .

Order labels using some well-founded ordering  $<$ .

Confluent if all local peaks completable to **locally decreasing diagrams**.

Local peak has form

$$s_1 \leftarrow_{\alpha} t \rightarrow_{\beta} s_2$$

Valley has form

$$s_1 \xrightarrow{\gamma_{\alpha}} \xrightarrow{\beta} \xrightarrow{(\gamma_{\alpha}) \cup (\gamma_{\beta})} \hat{t} \xleftarrow{(\gamma_{\alpha}) \cup (\gamma_{\beta})} \xleftarrow{\alpha} \xleftarrow{\gamma_{\beta}} s_2$$

where:

$$\gamma_{\alpha} = \{\alpha' \mid \alpha' < \alpha\}$$

$$\rightarrow_A = \bigcup_{\alpha \in A} \rightarrow_{\alpha}$$

# Confluence of Combined Reduction

**Main Theorem.**  $\rightarrow_{ca}$  is confluent on typable terms.

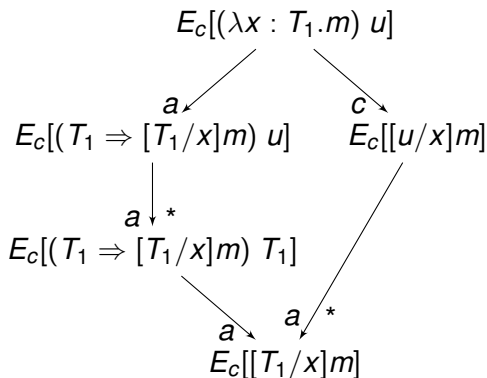
To prove, apply decreasing diagrams with  $a < c$ .

- $aa$ -peaks completable since  $\rightarrow_a$  has diamond property.
- No non-trivial  $cc$ -peaks, by determinism of  $\rightarrow_c$ .
- Two simple forms for  $ac$ -peaks, where  $m \rightarrow_a m'$ ,  $u \rightarrow_a u'$ .

$$\begin{array}{ccccc} E_c[(\lambda x : T. m') u] & \leftarrow_a & E_c[(\lambda x : T. m) u] & \rightarrow_c & E_c[[u/x]m] \\ E_c[(\lambda x : T. m) u'] & \leftarrow_a & E_c[(\lambda x : T. m) u] & \rightarrow_c & E_c[[u/x]m] \end{array}$$

- One more complex peak (next).

## Crucial Decreasing Diagram for Combined Reduction



$u \rightarrow_a^* T_1$ , since redex typable.

Locally decreasing, since  $c > a$ .



## Concluding Confluence

All local peaks completable to locally decreasing diagrams.

Therefore,  $\rightarrow_{ca}$  is confluent for typable terms.

Type Preservation for STLC established by rewriting techniques.

New proof method.

In the paper, apply to other example systems:

STLC+fix, STLC+ $\forall$ , Uniform-STC

For Uniform-STC, we apply APROVE and ACP for  $\rightarrow_a$ .

## Quick Look: STLC with Type Inference

Extend types to include meta-variables  $\alpha_j$ .

Now  $a(\beta)$  rule uses **narrowing** to instantiate  $\alpha_j$ :

$$\frac{\sigma \text{ is mgu}(T_1, T_2)}{E_a[(T_1 \Rightarrow m) T_2] \rightarrow_a \sigma(E_a[m])} a(\beta)$$

Again establish confluence for typable terms.

Conclude Type Preservation.

# Conclusion: A New Approach to Type Systems

Can use rewriting techniques to prove Type Preservation.

Proofs are different, qualitatively simpler, partly automatable.

A new agenda: redevelop Type Theory using rewriting approach.

- 1 Dependent types.
- 2 Curry-style systems.
  - ▶ Confluence fails.
  - ▶ Prove Type Preservation (rewriting form) directly.
- 3 Normalization proofs.
- 4 Soundness of symbolic simulation.
  - ▶ Natural operational formulation.
  - ▶ Soundness often not proved.

