Purifying Natural Deduction Using Sequent Calculus

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Verified Programming

Thesis

The ability to state and prove properties of code is the crucial missing technology in the evolution of software.

- Stronger guarantees => less monitoring => higher performance.
- Ability to trust software opens up new applications.
- Confirmed quality helps open source, app stores, etc.
- Verification is a tool we don't have.

The GURU Verified Programming Language

- Functional language
- Dependently typed programs
- General recursion
- Notation for theorems, proofs about programs
- Unaliased mutable state
- Resource management layer
- Type/Proof-checker, compiler to C
- No concurrency
- Aliasing for mutable state in progress

`www.guru-lang.org`

Practical Proof Theory

- How to prove your logic is consistent?
- Basic strategy:
  1. Identify subset of proofs which obviously are ok.
  2. Define rewrite rules to transform any proof to one in the ok form.
  3. Prove rules are (strongly or weakly) normalizing.
- By Curry-Howard isomorphism:
  - Proofs are λ-terms.
  - Proof normalization is β-reduction.
- Reducibility proofs (logical relations) are powerful, elegant.
- Do not work well with disjunctions, existentials.

Reducibility for Conjunction

Proof terms $p ::= (p_1, p_2) \mid p.1 \mid p.2$

$$\frac{\Gamma \vdash p_1 : \phi_1 \quad \Gamma \vdash p_2 : \phi_2}{\Gamma \vdash (p_1, p_2) : \phi_1 \wedge \phi_2} \wedge\text{I}$$

$$\frac{\Gamma \vdash p : \phi_1 \wedge \phi_2 \quad i \in \{1, 2\}}{\Gamma \vdash p.i : \phi_i} \wedge\text{E}$$

Reducibility is “hereditary normalization”, defined by eliminations.

- Red_ϕ is set of reducible terms of type ϕ.
- $p \in Red_b \;\Leftrightarrow\; SN(p)$, for base types b.
- $p \in Red_{\phi_1 \wedge \phi_2} \;\Leftrightarrow\; p.1 \in Red_{\phi_1}$ and $p.2 \in Red_{\phi_2}$.
- $p \in Red_{\phi_1 \rightarrow \phi_2} \;\Leftrightarrow\;$ forall $p' \in Red_{\phi_1}, (p\ p') \in Red_{\phi_2}$

What Goes Wrong with Disjunction

Proof terms $p ::= \langle 1, p\rangle \mid \langle 2, p\rangle \mid case(p)(x.p_1, x.p_2)$

$$\frac{\Gamma \vdash p : \phi_i \quad i \in \{1,2\}}{\Gamma \vdash \langle i, p\rangle : \phi_1 \wedge \phi_2} \vee\mathrm{I}$$

$$\frac{\Gamma \vdash p : \phi_1 \vee \phi_2 \quad \Gamma, x : \phi_1 \vdash p_1 : \psi \quad \Gamma, x : \phi_2 \vdash p_2 : \psi}{\Gamma \vdash case(p)(x.p_1, x.p_2) : \psi} \vee\mathrm{E}$$

Attempt to define reducibility fails:

$$p \in Red_{\phi_1 \vee \phi_2} \;\Leftrightarrow\; \text{forall } \psi,\; p_1, p_2 \in Red_\psi, case(p)(x.p_1, x.p_2) \in Red_\psi$$

Not legal to appeal to Red_ψ.

A Way Forward

- Problem with $\vee$E:
  - to use $p : \phi$, need $p' : \psi$, where ψ unrelated to ϕ.
  - breaks definition of reducibility.
- But compare sequent calculus rules:

$$\frac{\Gamma, \phi_1 \vdash \psi \quad \Gamma, \phi_2 \vdash \psi}{\Gamma, \phi_1 \vee \phi_2 \vdash \psi}\ \mathsf{L}\vee \qquad \frac{\Gamma, \phi_1, \phi_2 \vdash \psi}{\Gamma, \phi_1 \wedge \phi_2 \vdash \psi}\ \mathsf{L}\wedge$$

- Term assignment for sequent calculus is strange.

$$\frac{\Gamma, y : \phi_1, z : \phi_2 \vdash p : \psi}{\Gamma, x : \phi_1 \wedge \phi_2 \vdash [x.1/y, x.2/z]p : \psi}\ \mathsf{L}\wedge$$

- Limited by old view of “natural” deduction.

A Direct Term Assignment

- Left rules correspond to eliminations.
- Why insist that the context Γ holds just variables?
- Proposal:
  - Assign terms to sequent calculus directly.
  - Devise new terms for $\vee$E, $\exists$E.
  - Allow Γ to hold terms.

Elimination Rules

$$\frac{\Gamma, p.1 : \phi_1, p.2 : \phi_2 \vdash p' : \psi}{\Gamma, p : \phi_1 \wedge \phi_2 \vdash p' : \psi}\ \mathsf{L}\wedge$$

$$\frac{\begin{array}{c}\Gamma, p.(1) : \phi_1 \vdash p_1 : \psi \\ \Gamma, p.(2) : \phi_2 \vdash p_2 : \psi\end{array}}{\Gamma, p : \phi_1 \vee \phi_2 \vdash p_1 \,||\, p_2 : \psi}\ \mathsf{L}\vee$$

$$\frac{\Gamma, (p\ a) : [a/x]\phi \vdash p' : \psi}{\Gamma, p : \forall x.\phi \vdash p' : \psi}\ \mathsf{L}\forall$$

$$\frac{\Gamma, p!x : \phi \vdash p' : \psi \quad x \notin FV(\Gamma, \psi)}{\Gamma, p : \exists x.\phi \vdash \nu x.p' : \psi}\ \mathsf{L}\exists$$

$$\frac{}{p : \phi \vdash p : \phi}\ \mathsf{Ax}$$

$$\frac{\Gamma \vdash p_2 : \phi_2 \quad \Gamma, (p_1\ p_2) : \phi_1 \vdash p' : \psi}{\Gamma, p_1 : \phi_2 \rightarrow \phi_1 \vdash p' : \psi}\ \mathsf{L}\rightarrow$$

$$\frac{\Gamma \vdash p' : \psi}{\Gamma, p : \phi \vdash [p]p' : \psi}\ \mathsf{LW}$$

$$\frac{\Gamma, p : \phi, p : \phi \vdash p' : \psi}{\Gamma, p : \phi \vdash p' : \psi}\ \mathsf{LC}$$

Reduction

- We have separated logical terms ($p.(i)$) from structural ($p_1 \,||\, p_2$).
- Logical terms have β-reductions:

$$
\begin{array}{l}
(p_1, p_2).i \;\rightsquigarrow\; p_i \\
\langle i, p\rangle.(i) \;\rightsquigarrow\; t \\
\langle i, p\rangle.(3 - i) \;\rightsquigarrow\; abort
\end{array}
$$

- Structural terms have commuting conversions:

$$
\begin{array}{l}
(p_1 \,||\, p_2).i \;\rightsquigarrow\; (p_1.i) \,||\, (p_2.i) \\
abort \,||\, p \;\rightsquigarrow\; p
\end{array}
$$

- Simple unsound typing rules suffice for reducibility.

$$
\frac{\Gamma \vdash p : \phi_1 \vee \phi_2}{\Gamma \vdash p.i : \phi_i} \; \vee\mathrm{E}
$$

Towards Pure Natural Deduction

- Define natural deduction rules.

$$
\begin{array}{rcl}
S & ::= & \Gamma \vdash \Delta \mid S \,||\, S \\
\Delta & ::= & t_1 : \phi_1, \ldots, t_n : \phi_n
\end{array}
$$

- Example derivation:

$$
\dfrac{\dfrac{u : \phi_1 \vee \phi_2}{\begin{array}{ccc} u.(1) : \phi_1 & || & u.(2) : \phi_2 \\ \vdots & & \vdots \\ p_1 : \psi & || & p_2 : \psi \end{array}}}{p_1 \,||\, p_2 : \psi}
$$

- Completeness proved (open derivations $S \rhd S'$).
- Goal: Pure Natural Deduction.
  - All rules are either direct logical rules or structural.
  - Consistency proved by reducibility.
  - Decidable equational theory, including commuting conversions.