Programming and Proving with Dependently Typed Lambda Encodings

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Type theory based on lambda encodings

- Streamline the type theory
  - All data are lambda-encoded
  - No primitive constructors, pattern-matching
  - Simpler meta-theory for the type theory
- Expand its range with higher-order encodings
  - Inductive datatypes require (strict) positivity in Coq/Agda
  - With lambda-encodings, can go negative!

$$\textit{trm}\ t \quad ::= \quad x \mid t\ t' \mid \lambda x.t$$

encoded as

$$\textit{trm} \quad = \quad \forall X : \star.\ \underbrace{(X \to X \to X)}_{\textit{app}} \to \underbrace{((X \to X) \to X)}_{\textit{lam}} \to X$$

  - May enable simpler meta-theory for object languages
  - New range of tools to apply

Calculus of Dependent Lambda Eliminations (CDLE)

- Like Calculus of Constructions plus
  - functions with erased arguments (specificational, type is $\forall x : T.T'$)
  - constructor-constrained recursive types $\nu X : \kappa \,|\, \Theta.\ T$
  - lifting operator $\uparrow_L\ t$
- Semantics for types, consistency proof
- Realizability-inspired type assignment rules

$$\frac{\Gamma \vdash t : T \quad \Gamma \vdash t : [t/x]T'}{\Gamma \vdash t : \iota\, x{:}T.\, T'}$$

$$\frac{\Gamma \vdash t' : T \quad t =_\beta t'}{\Gamma \vdash t : T}$$

- Problem: how to implement?

From CDLE to Cedille

- First attempt: evidential typing $\Gamma \vdash e :: t : T$

  Unworkably painful!

- Insight: problematic rules used mainly for kinding ν-types
- Special-case treatment of those
- Dependent intersection types $\iota x : T . T'$ completely hidden
- Cedille implementation
  - New (wrote over winter break)
  - Around 2800 lines of Agda
  - 180-line grammar
  - around 300 lines of elisp

Datatypes in Cedille

- Can always use definitions as in System F_ω

```
CNat ⇐ ★ = ∀ X : ★ . (X → X) → X → X .
```

  - For these you get lifting
  - But no dependent eliminations/induction
  - Can be negative
  - Church-encoding only

- Can declare top-level recursive types

```
rec Nat | S : Nat → Nat , Z : Nat =
  ∀ P : Nat → ★ .
    (Π n : Nat . P n → P (S n)) → P Z → P self
  with
    S = λ n . Λ P . λ s . λ z . s n (n · P s z) ,
    Z = Λ P . λ s . λ z . z.
```

  - Lifting, dependent eliminations
  - Must be positive-recursive
  - All encodings supported (Church, Parigot, others)

Recursive types in Cedille

```
rec Nat | S : Nat → Nat , Z : Nat =
  ∀ P : Nat → ⋆ .
    (Π n : Nat . P n → P (S n)) → P Z → P self
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    S = λ n . Λ P . λ s . λ z . s n (n · P s z) ,
    Z = Λ P . λ s . λ z . z.
```

- `self` is implicitly ι-bound
- First declare constructors
- Then define them
- For definition $x = t$, type-check t under assumption $x = t$
- Afterwards, ι-type instantiated immediately on use

Rule induction

- Not all inductions are dependent eliminations(!)
  - Dependent elimination with x proves $P\ x$
  - If P does not depend on x, this is overkill
- PL meta-theory mostly uses *rule induction*

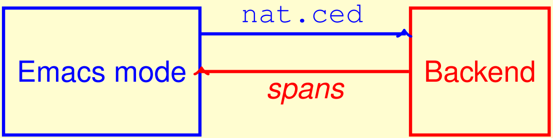
Theorem (Type Preservation)

If $\Gamma \vdash t : T$ *and* $t \rightsquigarrow t'$ *then* $\Gamma \vdash t' : T$.

- Can Church-encode the derivations
- But if you need inversion, should use Parigot

Emacs mode

- Cedille has an emacs mode for editing Cedille files
- Based on a generic structured-editing mode by Carl Olson



- A span is [*label*, *start-pos*, *end-pos*, *attributes*]
- Spans communicated in JSON
- Cedille sends <u>all</u> type information, in span attributes
- Monadic style for writing the backend (type checker)

Type checking

- Terms are annotated
  - $\Lambda x : \kappa.t$ for type abstraction
  - $t \cdot T$ for type instantiation
  - $\Lambda x : T.t$ for erased-term abstraction
  - $t\ -t'$ for erased-term application
- Use local type inference
  - Checking $t \Leftarrow T$
  - Synthesizing $t \Rightarrow T$
  - Can drop some annotations ($\lambda x.t$ instead of $\lambda x : T.t$)
- Conversion relation
  - Check if types are β-equivalent
  - All annotations erased from terms

Equality types

- $t \simeq t'$ means t is β-equivalent to t'
- Term constructs for equational reasoning
  - For $\epsilon\ t \Leftarrow t_1 \simeq t_2$, head-normalize t_1 and t_2 and check against t
  - For $\epsilon\ t \Rightarrow$, synthesize $t_1 \simeq t_2$ from t and head-normalize the sides
  - For $\rho\ t\ -\ t' \Leftarrow T$, synthesize $t_1 \simeq t_2$ from t and rewrite t_1 to t_2 in T before checking t'
  - For $\rho\ t\ -\ t' \Rightarrow$, synthesize $t_1 \simeq t_2$ from t and T from t', then rewrite t_1 to t_2 in T
  - ...
  - $\delta\ t \Leftarrow T$ succeeds if t synthesizes an obviously impossible equation
- Head normalization seems helpful for controlling reduction

A couple higher-order examples

1. Closedness preservation
   1. Define untyped λ-terms with free variables
   2. Define β-reduction
   3. Prove that reduction cannot introduce a free variable
2. Type preservation for STLC
   1. Define annotated terms, reduction
   2. Define typing <u>algorithm</u>
   3. Prove preservation

Cedille in action

Conclusion

- Cedille implementation of Calc. of Dep. Lambda Eliminations
  - Top-level datatype definitions with ctor-constrained rec. types
  - Emacs mode with structured editing
  - Equality types and rewriting
- Next direction: exploring higher-order encodings
  - Cedille is first dependent type theory supporting these
  - Could greatly simplify meta-programming/-proving
  - Uncharted territory...
    - ★ Non-dependent encodings (F_ω)
    - ★ Dependent ones ($\nu X : \kappa \,|\, \Theta.\, T$)
    - ★ Computing types from terms
    - ★ Algorithmic definitions

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Thank you!