Extending Unannotated System F with Positive-Recursive Types

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1 Introduction

In this note, we will extend unannotated System F with positive-recursive types $\mu X.T$, to obtain an unannotated type system F^{μ}. One way to think about a recursive type $\mu X.T$ is that it is a finite representation of an infinite type expression. For example, suppose we have a type like $\mu X.A \to X$. We could think of this as abbreviating the infinite type

$$A \to (A \to (A \to \cdots))$$

We won't actually use infinite types in either our type system or our metatheoretic analysis of that type system, but they can provide helpful intuition. For example, it is clear that if recursive types were thought of as denoting infinite type expressions, then the type $\mu X.T$ would always be equal to $[\mu X.T/X]T$. For the small example just considered: $\mu X.A \to X$ is equal to $A \to (\mu X.A \to X)$, which can be confirmed informally be thinking of the infinite expansion of each type.

This example type $\mu X.A \to X$ is not terribly useful, but we can use recursive types to assign types for Parigot-encoded datatypes. For example, recall that in the Parigot-encoding, the first few numbers are:

$$\begin{array}{rcl} 0 & = & \lambda s.\lambda z.z \\ 1 & = & \lambda s.\lambda z.s\ 0\ z \\ 2 & = & \lambda s.\lambda z.s\ 1\ (s\ 0\ z) \\ 3 & = & \lambda s.\lambda z.s\ 2\ (s\ 1\ (s\ 0\ z)) \end{array}$$

So the s variable has to take in a Nat as its first argument. But the type for s is part of the Nat type itself, and hence we need recursive types. The type for Parigot-encoded (unary) natural numbers is

$$\mu Nat.\ \forall X.(Nat \to (X \to X)) \to (X \to X)$$

1.1 Positivity

It is critical that in a recursive type $\mu X.T$, the type variable X occurs only *positively* in T. This means that X must occur in the domain type of an even number (possibly 0) of arrows. For example, in the type just above for Parigot-encoded natural numbers, the type variable Nat occurs in the domain type of this function type:

$$(Nat \to (X \to X))$$

And then that type expression (and hence the occurrence of Nat) occurs in the domain type of this function type:

$$(Nat \to (X \to X)) \to (X \to X)$$

So this occurrence of Nat (the only one in the body of the μ-type) occurs in the left part of two arrow types, and hence is positive.

$$\frac{\Gamma(x) = T}{\Gamma \vdash x : T} \qquad \frac{\Gamma, x : T \vdash t : T'}{\Gamma \vdash \lambda x.t : T \to T'} \qquad \frac{\Gamma \vdash t : T_1 \to T_2 \quad \Gamma \vdash t' : T_1}{\Gamma \vdash t\ t' : T_2}$$

$$\frac{\Gamma, X : \star \vdash t : T}{\Gamma \vdash t : \forall X.T} \qquad \frac{\Gamma \vdash t : \forall X.T \quad \Gamma \vdash T' : \star}{\Gamma \vdash t : [T'/X]T} \qquad \frac{\Gamma \vdash t : [\mu X.T/X]T \quad X \in^+ T}{\Gamma \vdash t : \mu X.T}$$

$$\frac{\Gamma \vdash t : \mu X.T \quad X \in^+ T}{\Gamma \vdash t : [\mu X.T/X]T}$$

Figure 1: Typing rules for F^μ

The positivity restriction is needed to ensure normalization. Without this restriction, we can actually assign the type $\mu X.X \to X$ to the non-normalizing term

$$(\lambda x.\, x\ x)\ (\lambda x.\, x\ x)$$

How is that done? First, we can assign that same type $\mu X.X \to X$ to $\lambda x.x\ x$ as follows. Since $\mu X.X \to X$ is equal to $(\mu X.X \to X) \to (\mu X.X \to X)$, by substituting the whole type expression in for X in $X \to X$, we may assume that the λ-bound variable x has type $\mu X.X \to X$. We must then show that $x\ x$ also has that type:

$$\cdot, x : (\mu X.X \to X) \vdash x\ x : (\mu X.X \to X)$$

But since as just observed, the type of x is equal to $(\mu X.X \to X) \to (\mu X.X \to X)$, we may apply x (with the type $(\mu X.X \to X) \to (\mu X.X \to X)$) to itself (with the type $\mu X.X \to X$). We have shown that $\lambda x.x\ x$ has type $\mu X.X \to X$. But by the same reasoning as when just now applying $x\ x$, we can then apply $\lambda x.x\ x$ to itself, with result type $\mu X.X \to X$.

We will see exactly where the requirement of positivity of X in T for $\mu X.T$ is needed, in the proof of Normalization below.

2 Syntax

term variables x		
type variables X		
terms t	$::=$	$x \mid \lambda x.t \mid t\ t'$
types T	$::=$	$X \mid T \to T' \mid \forall X.T \mid \mu X.T$

3 Typing

As for System F, we need typing contexts Γ, subject to the same implicit restriction we used before, of at most one declaration for each variable:

$$\textit{Typing context}\ \Gamma ::= \cdot \mid \Gamma, x : T \mid \Gamma, X : \star$$

The typing rules are in Figure 1, and kinding rules in Figure 2. To enforce positivity of X in T for $\mu X.T$, we use an extra judgement $x \in^p T$, where $p \in \{+, -\}$. This judgement is defined in Figure 3, where we also write $\bar{p}$ for the other polarity besides p (so $\bar{+} = -$ and $\bar{-} = +$). In the rules for concluding $X \in^p \forall Y.T$ and $X \in^p \mu Y.T$, we are requiring that Y is a different variable from X. The $\forall$- or μ-bound variable can always be implicitly renamed so that this requirement is met.

$$\frac{\Gamma(X) = \star}{\Gamma \vdash X : \star} \qquad \frac{\Gamma \vdash T_1 : \star \quad \Gamma \vdash T_2 : \star}{\Gamma \vdash T_1 \to T_2 : \star} \qquad \frac{\Gamma, X : \star \vdash T : \star}{\Gamma \vdash \forall X.T : \star} \qquad \frac{\Gamma, X : \star \vdash T : \star \quad X \in^+ T}{\Gamma \vdash \mu X.T : \star}$$

Figure 2: Kinding rules for F^μ

$$\frac{}{X \in^+ X} \qquad \frac{X \neq Y}{X \in^p Y} \qquad \frac{X \in^p T \quad X \neq Y}{X \in^p \forall Y.T} \qquad \frac{X \in^p T \quad X \neq Y}{X \in^p \mu Y.T} \qquad \frac{X \in^{\bar{p}} T_1 \quad X \in^p T_2}{X \in^p T_1 \to T_2}$$

Figure 3: Polarity of occurrences of type variables

4 Call-By-Name Normalization

The proof of call-by-name normalization for F^μ proceeds very similarly to the proof for unannotated System F. The basic setup is the same. We will define a semantics $[\![T]\!]_\rho$ for types T and functions ρ mapping the free type variables of T to reducibility candidates. As before, we denote the set of closed terms which normalize using call-by-name reduction as $\mathcal{N}$, and write $\leadsto$ for call-by-name reduction. A reducibility candidate R is again a set of terms satisfying:

- $R \subseteq \mathcal{N}$
- If $t \in R$ and $t' \leadsto t$, then $t' \in R$

The set of all reducibility candidates is again denoted $\mathcal{R}$. By the exact same argument as previously, we have:

Lemma 1 ($\mathcal{R}$ is a cpo)**.** *The set $\mathcal{R}$ ordered by subset forms a complete partial order, with greatest element $\mathcal{N}$ and greatest lower bound of a nonempty set of elements of $\mathcal{R}$ given by intersection.*

We may also easily observe that $\emptyset \in \mathcal{R}$.

The crucial new aspect of the semantics of types is that we must interpret $\mu X.T$. To do this, we will use the least fixed point of a monotonic function from $\mathcal{R}$ to $\mathcal{R}$. Since $\mathcal{R}$ is a cpo, such functions indeed have least fixed points. This is a powerful use of Lemma 1, and of the theory of least fixed points. Interpreting $\mu X.T$ as a least fixed point will ensure that $[\![\mu X.T]\!]_\rho$ equals $[\![[\mu X.T/X]T]\!]_\rho$, which will be needed to prove soundness of the typing rules involving μ-types.

We will use the following least fixed point theorem, which is standard for complete partial orders:

Theorem 2 (LFP)**.** *If f is a monotonic function from complete partial order $(X, \sqsubseteq, \sqcap)$ to itself, then the least fixed point of f is $\sqcap\{a \in X \mid f(a) \sqsubseteq a\}$.*

4.1 Semantics for types

Figure 4 gives the semantics $[\![T]\!]_\rho$ for types. The definitions for type variables, arrow types, and universal types are exactly as for System F. The new case is for μ-types. Here, we are following an approach where we try to give an interpretation for all types, not just those which are kindable. So we do not insist that $X \in^+ T$ in order to define an interpretation of $\mu X.T$ – although this is the only case we are really interested in for showing normalization for typable terms. We do need $X \in^+ T$ to show existence of the least fixed point (*lfp*) of the meta-level function

$$R \in \mathcal{R} \mapsto [\![T]\!]_{\rho[X \mapsto R]}$$

$$
\begin{array}{lcl}
[\![X]\!]_\rho & = & \rho(X) \\[2ex]
[\![T_1 \to T_2]\!]_\rho & = & \{t \in \mathcal{N} \mid \forall t' \in [\![T_1]\!]_\rho.\ t\ t' \in [\![T_2]\!]_\rho\} \\[2ex]
[\![\forall X.T]\!]_\rho & = & \bigcap\{[\![T]\!]_{\rho[X \mapsto R]} \mid R \in \mathcal{R}\} \\[2ex]
[\![\mu X.T]\!]_\rho & = & \bigcap\{R \ \in \mathcal{R} \mid [\![T]\!]_{\rho[X \mapsto R]} \subseteq R\}
\end{array}
$$

Figure 4: Reducibility semantics for types

This function takes a reducibility candidate R as input, and returns $[\![T]\!]_{\rho[X\mapsto R]}$. The crucial result is that if $X \in^+ T$, then this function is monotonic, and hence, since $\mathcal{R}$ is a cpo, has a least fixed point. But in defining the interpretation of types, it will be technically a little easier not to rely on kindability of the type being interpreted, nor try to prove that the function above is monotonic in the middle of the definition of the interpretation of types. So we state that the interpretation of a μ-type is the intersection of the set of reducibility candidates which are closed under the meta-level function displayed above, and then show later that when $X \in^+ T$, that function is monotonic and hence the intersection of that set of reducibility candidates is the least point of the function.

Lemma 3 (The semantics of types computes reducibility candidates)**.** *If $\rho(X)$ is defined for every free type variable of T, then $[\![T]\!]_\rho \in \mathcal{R}$.*

Proof. The proof is by induction on the structure of T. The cases for for type variables, arrow types, and universal types are exactly as for System F, so we will not repeat them. For $\mu X.T$, the definition makes $[\![\mu X.T]\!]_\rho$ the intersection of a set of reducibility candidates. Since $\mathcal{R}$ is a complete partial order, such an intersection is guaranteed to be an element of $\mathcal{R}$. □

Lemma 4 (Monotonicity)**.** *Suppose $R \subseteq R'$. If $X \in^+ T$, then for all ρ mapping all free type variables of T, we have $[\![T]\!]_{\rho[X\mapsto R]} \subseteq [\![T]\!]_{\rho[X\mapsto R']}$. And if $X \in^- T$, then similarly, for all such ρ, $[\![T]\!]_{\rho[X\mapsto R']} \subseteq [\![T]\!]_{\rho[X\mapsto R]}$.*

Proof. The proof is by mutual induction on the structure of the derivation of $X \in^+ T$.

<u>Case:</u>

$$\frac{}{X \in^+ X}$$

We have $[\![X]\!]_{\rho[X\mapsto R]} = R \subseteq R' = [\![X]\!]_{\rho[X\mapsto R']}$.

<u>Case:</u>

$$\frac{X \neq Y}{X \in^p Y}$$

In this case $[\![Y]\!]_{\rho[X\mapsto R]} = \rho(Y) = [\![Y]\!]_{\rho[X\mapsto R']}$.

<u>Case:</u>

$$\frac{X \in^p T \quad X \neq Y}{X \in^p \forall Y.T}$$

By the semantics of universal types, we have

$$
\begin{array}{lcl}
[\![\forall Y.T]\!]_{\rho[X\mapsto R]} & = & \bigcap_{R_a \in \mathcal{R}} [\![T]\!]_{\rho[Y\mapsto R_a][X\mapsto R]} \\
[\![\forall Y.T]\!]_{\rho[X\mapsto R']} & = & \bigcap_{R_a \in \mathcal{R}} [\![T]\!]_{\rho[Y\mapsto R_a][X\mapsto R']}
\end{array}
$$

It suffices to assume an arbitrary $R_a \in \mathcal{R}$, and relate $[\![T]\!]_{\rho[Y\mapsto R_a][X\mapsto R]}$ and $[\![T]\!]_{\rho[Y\mapsto R_a][X\mapsto R']}$ (according to the polarity p). By the IH, since $X \in^p T$, we have that the corresponding relationship holds: if $p = +$ then

$[\![T]\!]_{\rho[Y\mapsto R_a][X\mapsto R]} \subseteq [\![T]\!]_{\rho[Y\mapsto R_a][X\mapsto R']}$, and if $p = -$, then $[\![T]\!]_{\rho[Y\mapsto R_a][X\mapsto R']} \subseteq [\![T]\!]_{\rho[Y\mapsto R_a][X\mapsto R]}$. We are instantiating the variable assignment ρ in the IH with $\rho[Y \mapsto R_a]$.

Case:

$$\frac{X \in^{\bar{p}} T_1 \quad X \in^p T_2}{X \in^p T_1 \to T_2}$$

Suppose $p = +$. Then we may assume an arbitrary $t \in [\![T_1 \to T_2]\!]_{\rho[X\mapsto R]}$, and try to show $t \in [\![T_1 \to T_2]\!]_{\rho[X\mapsto R']}$. For the latter, it suffices by the semantics of arrow types to assume $t' \in [\![T_1]\!]_{\rho[X\mapsto R']}$, and show $t\ t' \in [\![T_2]\!]_{\rho[X\mapsto R']}$. By the IH, since $X \in^- T_1$, we know that $[\![T_1]\!]_{\rho[X\mapsto R']} \subseteq [\![T_1]\!]_{\rho[X\mapsto R]}$, and so $t' \in [\![T_1]\!]_{\rho[X\mapsto R]}$. This means that we have $t\ t' \in [\![T_1 \to T_2]\!]_{\rho[X\mapsto R]}$ by the semantics of arrow types, since we are assuming $t \in [\![T_1 \to T_2]\!]_{\rho[X\mapsto R]}$. Now we may apply the IH again, since $X \in^+ T_2$, to deduce that $[\![T_2]\!]_{\rho[X\mapsto R]} \subseteq [\![T_2]\!]_{\rho[X\mapsto R']}$. This gives us $t\ t' \in [\![T_1 \to T_2]\!]_{\rho[X\mapsto R']}$, as required. The case where $p = -$ is dual to this, so we omit it.

Case:

$$\frac{X \in^p T \quad X \neq Y}{X \in^p \mu Y.T}$$

For concreteness, suppose $p = +$ (the case for $p = -$ is similar). We must show

$$\bigcap\{R'' \in \mathcal{R} \mid [\![T]\!]_{\rho[X\mapsto R][Y\mapsto R'']} \subseteq R''\} \quad \subseteq \quad \bigcap\{R'' \in \mathcal{R} \mid [\![T]\!]_{\rho[X\mapsto R'][Y\mapsto R'']} \subseteq R''\}$$

To do this, assume an arbitrary t such that $\forall R'' \in \mathcal{R}.\ ([\![T]\!]_{\rho[X\mapsto R][Y\mapsto R'']} \subseteq R'') \ \to\ t \in R''$. It suffices then to show $\forall R'' \in \mathcal{R}.\ ([\![T]\!]_{\rho[X\mapsto R'][Y\mapsto R'']} \subseteq R'') \ \to\ t \in R''$. For the latter, assume an arbitrary $R'' \in \mathcal{R}$ with $[\![T]\!]_{\rho[X\mapsto R'][Y\mapsto R'']} \subseteq R''$. We must now show $t \in R''$. By the IH, we have

$$[\![T]\!]_{\rho[X\mapsto R'][Y\mapsto R'']} \ \subseteq\ [\![T]\!]_{\rho[X\mapsto R'][Y\mapsto R'']}$$

And so by transitivity of the subset relation, we have $[\![T]\!]_{\rho[X\mapsto R'][Y\mapsto R'']} \subseteq R''$. So we may instantiate the assumption we made about t when we introduced it, to conclude $t \in R''$ as required. $\square$

4.2 Soundness of typing

We claim this lemma without proof, as we did for System F:

Lemma 5. $[\![[T'/X]T]\!]_\rho = [\![T]\!]_{\rho[X\mapsto T']}$

Now using the same definition of $[\![\Gamma]\!]$ as we did for System F, we can prove:

Theorem 6 (Soundness of typing rules with respect to the semantics)**.** *If $\Gamma \vdash t : T$, then for all $(\sigma, \rho) \in [\![\Gamma]\!]$, we have $\sigma t \in [\![T]\!]_\rho$.*

Proof. The proof is by induction on the structure of the typing derivation. All cases are exactly as for System F, except for the new cases, for μ-types:

Case:

$$\frac{\Gamma \vdash t : [\mu X.T/X]T \quad X \in^+ T}{\Gamma \vdash t : \mu X.T}$$

By the IH, we have $\sigma t \in [\![[\mu X.T/X]T]\!]_\rho$. By Lemma 5, this gives us $\sigma t \in [\![T]\!]_{\rho[X\mapsto[\![\mu X.T]\!]_\rho]}$. By Lemma 4, the (meta-level) function $R \in \mathcal{R} \mapsto [\![T]\!]_{\rho[X\mapsto R]}$ is monotonic from $\mathcal{R}$ to $\mathcal{R}$, because $X \in^+ T$. So by the semantics of μ-types and Theorem 2, $[\![\mu X.T]\!]_\rho$ is the least fixed point of the function $R \in \mathcal{R} \mapsto [\![T]\!]_{\rho[X\mapsto R]}$. Since it is a fixed point, this means that

$$[\![\mu X.T]\!]_\rho = [\![T]\!]_{\rho[X\mapsto[\![\mu X.T]\!]_\rho]}$$

But we already observed that we have σt in the set described by the right-hand side of this equation. So this means we have $\sigma t \in [\![\mu X.T]\!]_\rho$ (the left-hand side), as required.

Case:

$$\frac{\Gamma \vdash t : \mu X.T \quad X \in^+ T}{\Gamma \vdash t : [\mu X.T/X]T}$$

By the IH, we have $\sigma t \in [\![\mu X.T]\!]_\rho$. As in the previous case, we may apply Lemma 4 and Theorem 2 to deduce that $[\![\mu X.T]\!]_\rho$ equals

$$\mathit{lfp}(R \in \mathcal{R} \mapsto [\![T]\!]_{\rho[X \mapsto R]})$$

So we have

$$[\![\mu X.T]\!]_\rho = [\![T]\!]_{\rho[X \mapsto [\![\mu X.T]\!]_\rho]}$$

And by similar reasoning as in the previous case, the right-hand side of this equation equals $[\![[\mu X.T/X]T]\!]_\rho$, giving us the required conclusion.

□

Corollary 7 (Normalization)**.** *If* Γ *declares no term variables and* $\Gamma \vdash t : T$ *then* t *is call-by-name normalizing.*

Proof. The proof is exactly as for System F, making use here of Theorem 6. □