

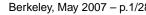
The Impact of Craig's Interpolation Theorem

in Computer Science

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The Role of Logic in Computer Science



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It provides formal foundations for

- 6 Programming languages
- 6 Relational databases
- 6 Computational complexity
- 6 Hardware design and validation
- 6 Formal methods in software engineering
- 6 Artificial Intelligence

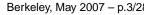
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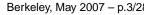


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- 6 together with compactness, is considered a crucial property of any new logic for CS
- 6 comes up in any formal method based on modular decomposition of complex systems



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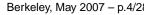
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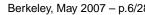


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Craig's Interpolation: If φ_1 and φ_2 are inconsistent, there is a φ in their shared language such that

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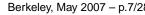
 $\varphi_1 \models \psi$ and $\psi \land \varphi_2$ is inconsistent.

Intuitively,

- 6 ψ is an abstraction of φ_1 from the viewpoint of φ_2 ;
- 6 ψ summarizes and translates in the shared language why φ_1 is inconsistent with φ_2 .



Part I: Craig Interpolation for Prover Combinations





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Some relevant theories in SMT

- 6 Equality with "Uninterpreted Function Symbols"
- 6 Linear Arithmetic (Real and Integer)
- 6 Arrays (i.e., updatable maps)
- 6 Bit vectors
- 6 Finite trees

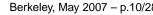


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Problem: In practice, we often need to deal with *mixed* formulas in $\mathcal{L}^{\Sigma_1 \cup \cdots \cup \Sigma_n}$ modulo a *combined theory* $T_1 \cup \cdots \cup T_n$.





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Problem: In practice, we often need to deal with *mixed* formulas in $\mathcal{L}^{\Sigma_1 \cup \cdots \cup \Sigma_n}$ modulo a *combined theory* $T_1 \cup \cdots \cup T_n$.

In that case, it helps if we can

combine modularly decision procedures for the individual T_1, \ldots, T_n into a decision procedure for $T_1 \cup \cdots \cup T_n$.

For i = 1, 2,

- 6 let T_i a first-order theory of signature Σ_i and
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the $(T_1 \cup T_2)$ -satisfiability of a formula $\varphi \in \mathcal{L}^{\Sigma_1 \cup \Sigma_2}$ is effectively reducible to the $(T_1 \cup T_2)$ -satisfiability of formulas of the form $\varphi_1 \wedge \varphi_2$ with $\varphi_i \in \mathcal{L}^{\Sigma_i}$ for i = 1, 2.

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Observation: For purifiable languages, $(T_1 \cup T_2)$ -satisfiability is at heart an interpolation problem.

Combined Satisfiability as Interpolation

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iff, by an application of Craig's interpolation theorem, there is a $(\Sigma_1 \cap \Sigma_2)$ -formula $\varphi(\mathbf{x})$ with $\mathbf{x} = \mathbf{x}_1 \cap \mathbf{x}_2$ s.t. $T_1, \varphi_1 \models \varphi$ and $T_2, \varphi_2, \varphi \models \bot$ For i = 1, 2, let T_i -be a Σ_i -theory and $\varphi_i[\mathbf{x}_i]$ a Σ_i -formula.

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The problem then is "just" computing the interpolant φ .

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All existing combination methods are in essence ways to compute φ , possibly incrementally, in finite time, without building a direct proof that $T_1, \varphi_1, T_2, \varphi_2 \models \bot$

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Historical note: The original correctness proof of the foremost combination method for SMT (Nelson & Oppen, 1979) relies directly on Craig's interpolation theorem.



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An Effectively Purifiable Language

The class of quantifier-free formulas is effectively purifiable for any Σ_1 and Σ_2 :

Given a quantifier-free $(\Sigma_1 \cup \Sigma_2)$ -formula φ

we can compute Σ_1 -qffs $\varphi_1^1 \dots \varphi_1^n$ and Σ_2 -qffs $\varphi_2^1 \dots \varphi_2^n$ s.t.

for every $(\Sigma_1 \cup \Sigma_2)$ -structure \mathcal{A} ,

 φ is satisfiable in \mathcal{A} iff $\varphi_1^j \wedge \varphi_2^j$ is satisfiable in \mathcal{A} for some j.



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Let's focus then on quantifier-free formulas.

For simplicity, but wlog, let's consider only combined satisfiability problems of the form

$\Gamma_1 \cup \Gamma_2$

where each Γ_i is a finite set of Σ_i -*literals* (i.e., atomic formulas and negated atomic formulas)

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 ψ_1, \ldots, ψ_n is an *interpolation chain* if for each $k = 1, \ldots, m$ there is an $i \in \{1, 2\}$ s.t.

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Under the right conditions:

- 1. $\Gamma_1 \cup \Gamma_2$ is $(T_1 \cup T_2)$ -unsatisfiable iff there is an interpolation chain ψ_1, \ldots, ψ_m with $\psi_n = \bot$, and
- 2. each ψ_i is a disjunction of atoms and is computable using one of the decision procedures for T_1 and T_2 .

Sufficient conditions on T_1 and T_2 (Ghilardi, 2005)

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Nelson-Oppen Method: $\Sigma_0 = \emptyset$ and each T_i is stably infinite.

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But others are **not**:

- 6 Theories of a finite structure.
- 5 Theories with models of bounded cardinality.
- Some equational/Horn theories.



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- 6 The trick is to require the decision procedures to also exchange finite-cardinality constraints.
- On these extensions are still instances of Craig interpolation.
- 6 However, they now consider interpolation chains that also include quantified formulas like

$$\forall x, y, z. \ x = y \lor x = z$$



SMT provers based on some variant of the Nelson-Oppen method are widely used in academia and industry.



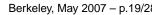
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6 The generalized results by Ghilardi have several additional applications.

For instance, they can be used in the combination of modals logics.



Part II: Craig Interpolation in Model Checking





Software or hardware systems can be often modeled as state transition systems $\mathcal{M} = (S, I, R, L)$ where

- \circ S is a set of states
- 6 $I \subseteq S$ is a set of *initial states*
- 6 $R \subseteq S \times S$ is a total *transition relation*
- 6 $L: S \rightarrow 2^{At}$ is a *labelling function* into sets of atomic formulas in some base logic



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Note: \mathcal{M} is a Kripke model (in the sense modal logic).



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Most system correctness properties can be expressed as a *safety* property for a suitable model \mathcal{M} :

 \mathcal{M} is *safe* wrt a property ψ if no state R-reachable from an initial state satisfies ψ .



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Model checking is one of the most successful areas of formal verification.

Model checking technologies are now routinely used in industry.

Symbolic Model Checking

A model $\mathcal{M} = (S, I, R, L:S \rightarrow 2^{At})$ can be expressed symbolically by fixing a set X of variables and a first-order Σ -structure \mathcal{A} with universe A.

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Then:

- 6 Every state $\sigma \in S$ is a mapping in $[X \to A]$
- 6 At is a set of atomic Σ -formulas over X
- 6 *I* is characterized by a qff $\varphi_I[\mathbf{x}]$ s.t. $\sigma \in I$ iff $\mathcal{A} \models \varphi_I[\sigma]$
- 6 *R* is characterized by a qff $\varphi_R[\mathbf{x}, \mathbf{x}']$ such that $(\sigma, \sigma') \in R$ iff $\mathcal{A} \models \varphi_R[\sigma, \sigma']$

Notation: if $\mathbf{x} = x_1, \ldots, x_n$ then $\psi[\sigma] = \psi[\sigma(x_1), \ldots, \sigma(x_n)]$



6 A state σ is *reachable (in k steps)* iff there is a sequence of states $\sigma_0, \ldots, \sigma_k = \sigma$ such that

 $\mathcal{A} \models \varphi_I[\sigma_0] \land \varphi_R[\sigma_0, \sigma_1] \land \dots \land \varphi_R[\sigma_{k-1}, \sigma_k]$

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Observation: \mathcal{M} is safe wrt ψ iff ψ is **not** reachable from φ_I iff

$$\varphi_I[\mathbf{x}_0] \wedge \varphi_R[\mathbf{x}_0, \mathbf{x}_1] \wedge \cdots \wedge \varphi_R[\mathbf{x}_{k-1}, \mathbf{x}_k] \wedge \psi[\mathbf{x}_k]$$

is unsatisfiable in \mathcal{A} for all $k \geq 0$.



6 For a large class of systems \mathcal{M} , we can compute from φ_I and φ_R the *strongest inductive invariant* φ_{IR} for \mathcal{M} :



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- 6 This can be completely automated if the satisfiability in \mathcal{A} of qffs is decidable.
- **6 Problem:** Computing φ_{IR} can be very expensive.
- 6 **Good news: Craig interpolation** can be used to reduce this cost.

When φ_{IR} is computable it is because it is the least fix point of an *image* operator $Img: QFF \rightarrow QFF$ where

6 $Img(\varphi[\mathbf{x}])$ is the strongest (wrt $\models_{\mathcal{A}}$, entailment in \mathcal{A}) qff $\varphi_{p}[\mathbf{x}]$ such that

$$\varphi[\mathbf{x}] \land \varphi_R[\mathbf{x}, \mathbf{x}'] \models_{\mathcal{A}} \varphi_P[\mathbf{x}']$$

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However, Img might be much stronger than needed for proving that a property ψ is unreachable.

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Idea (McMillan, 2003):

use interpolation to compute for each $i \ge 0$ an *adequate over-approximation* $\hat{\varphi}^i$ of φ^i wrt ψ

Let k > 0, $\hat{\varphi}^0 = \varphi_I[\mathbf{x}]$

Base Case) Let:

$$\Gamma_1 = \hat{\varphi}^0[\mathbf{x}_0] \wedge \varphi_R[\mathbf{x}_0, \mathbf{x}_1]$$

$$\Gamma_2 = \varphi_R[\mathbf{x}_1, \mathbf{x}_2] \wedge \cdots \wedge \varphi_R[\mathbf{x}_{k-1}, \mathbf{x}_k] \wedge (\psi[\mathbf{x}_1] \vee \cdots \vee \psi[\mathbf{x}_k])$$

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6 If $\Gamma_1 \wedge \Gamma_2$ is satisfiable in \mathcal{A} , we are done:

 ψ is reachable from φ_I in 1 to k steps.

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6 If $\Gamma_1 \wedge \Gamma_2$ is unsatisfiable in \mathcal{A} , compute an interpolant $\Gamma[\mathbf{x}_1]$ (wrt to $\models_{\mathcal{A}}$).

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- o Γ[x] is an adequate over-approximation of $Img(\varphi^0)$: $\Gamma_1 \models_{\mathcal{A}} \Gamma[x_1] \implies$ every state reachable from φ_I is in Γ $\Gamma \land \Gamma_2 \models_{\mathcal{A}} \bot \implies$ no state in Γ leads to ψ within k steps

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Assume we have computed $\hat{\varphi}^i$ for i > 0. **Step case)** Let

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6 Let
$$\hat{\varphi}^{i+1} = \hat{\varphi}^i[\mathbf{x}] \vee \Gamma[\mathbf{x}]$$

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- 6 If $\Gamma_1 \wedge \Gamma_2$ is satisfiable in \mathcal{A} , ψ is reachable from φ_I in i+1 to i+k steps in the overapproximated closure of φ_R

So, the satisfying paths of states might not be paths in the original system \mathcal{M} .

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5 Then, increase k by 1 and restart the whole process.



Thank you

