Solutions to Homework 3

22C:044 Algorithms, Fall 2000

1(a)

$$
\begin{aligned}
T(n) &= T(n-1) + (2n-3) \\
&= T(n-2) + [2(n-1)-3] + (2n-3) \\
&= T(n-3) + [2(n-2)-3] + [2(n-1)-3] + (2n-3) \\
&\vdots \\
&= T(n-i) + [2(n-i+1)-3] + [2(n-i+2)-3] + \ldots + (2n-3)
\end{aligned}
$$

If we iterate $i = n-1$ times we get

$$T(n) = T(1)+1+3+5+\ldots+(2n-3) = T(1)+(n-1)(2n-3+1)/2 = T(1)+(n-1)^2.$$

So we have $T(n) = \Theta(n^2)$.

The substitution: Let us prove using mathematical induction that $T(n) = T(1) + (n-1)^2$ for every n. (Notice that I can use the exact solution I got from the iteration method.)

1° The base $n = 1$ is correct: $T(n) = T(1) = T(1) + (n-1)^2$.

2° The inductive step: Assume that the claim is true for $n-1$, that is, assume that $T(n-1) = T(1) + (n-2)^2$. Then the claim is true for n as well:

$$T(n) = T(n-1)+(2n-3) = T(1)+(n-2)^2+(2n-3) = T(1)+(n-1)^2.$$

1(b) In master method $a = 1$ and $b = 4$. Because $\log_b a = 0$ we compare $n^0 = 1$ and $f(n) = \sqrt{n} + 1$. This is case 3 of the master theorem. We still have to verify the regularity condition

$$af(n/b) \le cf(n)$$

for some constant $c < 1$. In our case the regularity condition is:

$$\sqrt{n/4} + 1 \le c(\sqrt{n} + 1).$$

This is equivalent to

$$c \geq 1/2 + 1/(2\sqrt{n} + 2).$$

For all $n \geq 1$ the right-hand-side is at most $3/4$ so any constant c between $3/4$ and 1 will satisfy the regularity condition.

The master theorem gives the solution $T(n) = \Theta(\sqrt{n} + 1) = \Theta(\sqrt{n})$.

2 The random experiment involves $2n$ coin flips, n by each professor. Each elementary event has probability $0.5^{2n} = 1/4^n$. As pointed out in the question, the event that both professors get the same number of heads is the same as the event that total number of successful coin tosses is n. Each toss is either successful or unsuccessful with equal probability $1/2$. There are

$$\binom{2n}{n}$$

sequences of $2n$ tosses containing exactly n successes, and each sequence has the same probability $1/4^n$. The total probability is therefore

$$\binom{2n}{n} / 4^n.$$

On the other hand, the probability that professor R tosses exactly k heads is

$$\binom{n}{k} / 2^n,$$

so the probability that both professors toss exactly k heads is $\binom{n}{k}^2 / 4^n$. Summing up over all values of k from 0 to n we get the total probability of the professors getting the same number of heads:

$$\sum_{k=0}^{n} \binom{n}{k}^2 / 4^n.$$

This is of course the same as the probability calculated in the first part of the problem. Therefore,

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$$

3(a) The probability is $1/n$.

3(b) If line 5 is executed then $A[i]$ is the largest element among all $A[j]$, $1 \leq j \leq i$.

3(c) The probability is $1/i$.

3(d)

$$\begin{aligned} E[s_i] &= 0 \times Pr\{\text{"line 5 is not executed on the } i\text{'th iteration"}\} + \\ & \quad 1 \times Pr\{\text{"line 5 is executed on the } i\text{'th iteration"}\} \\ &= 1/i. \end{aligned}$$

3(e) $E[s_1 + s_2 + \ldots + s_n] = E[s_1] + E[s_2] + \ldots + E[s_n] = 1/1 + 1/2 + \ldots + 1/n$. The last sum S is known to be $\Theta(\lg n)$. To see that, recall from calculus that the integral of the function $1/x$ is $\ln x$. Therefore, the area below the curve $y = 1/x$ from $x = 1$ till $x = n$ is $A = \ln n$. On the other hand, upper and lower Riemann sums give

$$1/2 + 1/3 + 1/4 + \ldots + 1/n < A < 1/1 + 1/2 + 1/3 + \ldots + 1/(n-1).$$

The rightmost and leftmost expressions are $S - 1$ and $S - 1/n$, respectively, where $S = 1/1 + 1/2 + 1/3 + \ldots + 1/n$ is the sum we want to evaluate. So we have $S - 1 < \ln n < S - 1/n < S$, that is, $\ln n < S < \ln n + 1$.

4 7.1-1: The minimum number of elements is 2^h, and the maximum number of elements is $2^{h+1} - 1$.

7.1-4: Any leaf can be the smallest element.

7.1-5: Yes. The children of each node come later in the array, and are therefore no greater than the node itself.

7.3-1: